

# SURJECTIVITY OF MEAN VALUE OPERATORS ON NONCOMPACT SYMMETRIC SPACES

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**ABSTRACT.** Let  $X = G/K$  be a symmetric space of the non-compact type. We prove that the mean value operator over translated  $K$ -orbits of a fixed point is surjective on the space of smooth functions on  $X$  if  $X$  is either complex or of rank one. For higher rank spaces it is shown that the same statement is true for points in an appropriate Weyl subchamber.

## 1. INTRODUCTION

Let  $X = G/K$  be a symmetric space of the non-compact type. We will investigate the question of surjectivity of convolution operators on the space of smooth functions on  $X$ . Let  $\mathcal{E}(X)$  denote the space of smooth functions on  $X$  equipped with the topology of uniform convergence of all derivatives on compact sets. We will show, in particular, that for any  $y$  in  $X$  the mean value operator

$$M^y f(x) = \int_K f(gk \cdot y) dk \quad (x = g \cdot o \in X),$$

is a surjective linear operator on  $\mathcal{E}(X)$  if  $X$  is either complex or of rank one. The mean value operator above can be realized as a convolution operator with a  $K$ -invariant distribution of compact support. This allows us to transfer the problem to Euclidean space via the Abel transform, and in Euclidean space we can apply conditions on the convolution kernel which were previously obtained by Ehrenpreis ([Ehr60]) and Hörmander ([Hör05]). Furthermore, our approach allows us to conclude that  $G$ -invariant differential operators are surjective on smooth functions on symmetric spaces, which is one of the main results in [Hel73]. (See also [Ehr54] or [Mal54] for the corresponding result on Euclidean space.)

Mean value operators can also be thought of as Radon transforms related to double fibrations. (See [Hel08], Ch. II, §3). Some of the principal problems are to determine the kernels, ranges, and the mapping properties of these transforms and their duals on spaces of functions and distributions. For example, the dual classical Radon transform was shown to be surjective on the space of smooth functions by Hertle in 1984 ([Her84]). A similar result was proved by Helgason for the dual horocycle Radon transform on symmetric spaces. (See [Hel83] or [Hel08], Ch. IV, Corollary 2.5.)

Mean value operators are essentially self-dual integral transforms, so questions pertaining to these transforms and their duals coincide. The surjectivity of spherical mean value operators on Euclidean space as well as the hyperbolic space  $\mathbb{H}^3$  was recently proved in the thesis of K. Lim [Lim12]. The idea of using Ehrenpreis and Hörmander estimates in our work originates from the thesis by K. Lim.

## 2. THE EHRENPREIS AND HÖRMANDER CRITERIA

Throughout this paper we will use the following standard spaces on a smooth manifold  $\mathcal{M}$ . The space  $\mathcal{E}(\mathcal{M})$  denotes the space of smooth functions on  $\mathcal{M}$  equipped with the topology of uniform convergence of all derivatives on every compact subset of  $\mathcal{M}$ . The space  $\mathcal{D}(\mathcal{M})$  denotes

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the subspace of functions in  $\mathcal{E}(\mathcal{M})$  which are compactly supported. The dual  $\mathcal{D}'(\mathcal{M})$  is called the space of distributions on  $\mathcal{M}$ , and the dual  $\mathcal{E}'(\mathcal{M})$  is the space of compactly supported distributions. The dual spaces  $\mathcal{D}'(\mathcal{M})$  and  $\mathcal{E}'(\mathcal{M})$  are (if not stated otherwise) equipped with the weak\* topology.

In this section, we consider the convolution operator on  $\mathbb{R}^n$  with a given distribution of compact support. In particular, for a distribution  $\mu \in \mathcal{E}'(\mathbb{R}^n)$ , consider the convolution operator  $c_\mu : \mathcal{E}(\mathbb{R}^n) \rightarrow \mathcal{E}(\mathbb{R}^n)$  given by

$$(2.1) \quad c_\mu(f) = f * \mu.$$

Since  $\mu$  has compact support, the Fourier-Laplace transform  $\mu^*$  is a holomorphic function on  $\mathbb{C}^n$ . A complete description of the Fourier transforms of compactly supported distributions is of course provided by the Paley-Wiener Theorem. (See, for instance, Theorem 7.3.1 in [Hör03], where it is formulated in terms of support functions.)

The operator  $c_\mu$  is, in general, not injective on  $\mathcal{E}(\mathbb{R}^n)$ . In fact suppose that  $\zeta \in \mathbb{C}^n$  is a zero of the holomorphic function  $\mu^*$ . Then if  $f(x) = e^{i\langle \zeta, x \rangle}$ , we would have  $c_\mu(f) = 0$ . However, if  $\mu \neq 0$ , then  $c_\mu$  is injective as an operator on  $\mathcal{E}'(\mathbb{R}^n)$ , as is easily seen by taking Fourier transforms.

On the other hand,  $c_\mu : \mathcal{E}(\mathbb{R}^n) \rightarrow \mathcal{E}(\mathbb{R}^n)$  is often surjective, as for instance when  $c_\mu$  happens to be a constant coefficient differential operator [Ehr54, Theorem 10]. In the theorem below, we collect conditions provided in Hörmander's text ([Hör05], Theorem 16.3.10, Definition 16.3.12, and Theorem 16.5.7) which are equivalent to the surjectivity of  $c_\mu$ . In order to formulate the theorem we need a definition:

**Definition 2.1.** We will say that a function  $u : \mathbb{C}^n \rightarrow \mathbb{C}$  is *slowly decreasing* if there is a constant  $A > 0$  such that

$$\sup\{|u(\zeta)| : \zeta \in \mathbb{C}^n, \|\zeta - \xi\| \leq A \log(2 + \|\xi\|)\} \geq (A + \|\xi\|)^{-A},$$

for all  $\xi \in \mathbb{R}^n$ .

This is easily shown to be equivalent to the following seemingly more flexible condition: the function  $u : \mathbb{C}^n \rightarrow \mathbb{C}$  is slowly decreasing if and only if there are positive constants  $A, B, C$ , and  $D$  such that

$$(2.2) \quad \sup\{|u(\zeta)| : \zeta \in \mathbb{C}^n, \|\zeta - \xi\| \leq A \log(2 + \|\xi\|)\} \geq B(C + \|\xi\|)^{-D}$$

for all  $\xi \in \mathbb{R}^n$ .

In what follows, let  $\mu^\vee$  be the distribution  $\mu^\vee(f) = \mu(f^\vee)$ , for  $f \in \mathcal{E}(X)$ , where  $f^\vee(x) = f(-x)$ .

**Theorem 2.2.** (Ehrenpreis [Ehr60], Hörmander [Hör05]) *Let  $\mu \in \mathcal{E}'(\mathbb{R}^n)$ . Then the following conditions on  $\mu$  are equivalent.*

- (i) *The convolution operator  $c_\mu : \mathcal{E}(\mathbb{R}^n) \rightarrow \mathcal{E}(\mathbb{R}^n)$  is surjective.*
- (ii) *The Fourier transform  $\mu^*$  is slowly decreasing.*
- (iii) *For any  $T \in \mathcal{E}'(\mathbb{R}^n)$  such that  $T^*/\mu^*$  is an entire function on  $\mathbb{C}^n$ , there is an  $S \in \mathcal{E}'(\mathbb{R}^n)$  such that  $S^* = T^*/\mu^*$ .*
- (iv) *The convolution operator  $c_{\mu^\vee} : \mathcal{E}'(\mathbb{R}^n) \rightarrow \mathcal{E}'(\mathbb{R}^n)$  has weak\* closed range.*

In the aforementioned reference, there are additional conditions on  $\mu$  in case the domain and range of  $c_\mu$  are required to have support in certain subsets of  $\mathbb{R}^n$ , but we will not need them here since they are trivially satisfied when these subsets equal  $\mathbb{R}^n$ .

Following Hörmander, we will say that a distribution  $\mu \in \mathcal{E}'(\mathbb{R}^n)$  is *invertible* provided that it satisfies any of the equivalent conditions in Theorem 2.2.

Since  $c_{\mu^\vee} : \mathcal{E}'(\mathbb{R}^n) \rightarrow \mathcal{E}'(\mathbb{R}^n)$  is injective if  $\mu \neq 0$ , the equivalence of Conditions (i) and (iv) above is a special case of the following general fact about continuous linear mappings on Frechét spaces. (In [Hör05], Theorem 16.5.7, it is used to prove that (iv) implies (i).)

**Theorem 2.3.** *Let  $E$  and  $F$  be Frechét spaces. A continuous linear map  $\Phi: E \rightarrow F$  is surjective if and only if its adjoint  $\Phi^*: F' \rightarrow E'$  is injective and has a weak\* closed range in  $E'$ .*

For a proof of Theorem 2.3, see Theorem 7.7, Ch. IV in Schaefer's book [Sch71]. See also Theorem 3.7, Ch. I of [Hel08] for a generalization. We note that it is a straightforward consequence of the Hahn-Banach Theorem that a subspace of  $E'$  is closed in the weak\* topology of  $E'$  if and only if it is closed in the strong topology of  $E'$ .

Note that while the other conditions in Theorem 2.2 are mapping conditions, Condition (ii) is a condition that is in theory testable by computation. According to [Mal56], if the ratio  $T^*/\mu^*$  is an entire function on  $\mathbb{C}^n$ , then it is of exponential type. In order for this ratio to be the Fourier transform of a compactly supported distribution, it must be of slow (i.e., polynomial) growth in  $\mathbb{R}^n$ , and the slow decrease condition (ii) on  $\mu^*$  is equivalent to this.

Finally, we note that if  $\mu$  is invertible, then so is  $\mu^\vee$ , and the (closed) range  $c_\mu(\mathcal{E}'(\mathbb{R}^n))$  is given precisely by the set

$$(2.3) \quad \{S \in \mathcal{E}'(\mathbb{R}^n) : S^*(\zeta)/\mu^*(\zeta) \text{ is an entire function on } \mathbb{C}^n\}.$$

We will need the following refinement of Condition (iii), which can be found in the proof of Theorem 16.3.10 in [Hör05].

**Proposition 2.4.** *Suppose that  $u: \mathbb{C}^n \rightarrow \mathbb{C}$  is slowly decreasing and holomorphic. For every triple  $(C, R, N) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{Z}^+$ , there is a triple  $(C', R', N') \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{Z}^+$  such that whenever  $v: \mathbb{C}^n \rightarrow \mathbb{C}$  is holomorphic and satisfies*

$$|v(\zeta)| \leq C (1 + \|\zeta\|)^N e^{R\|Im\zeta\|}$$

for all  $\zeta \in \mathbb{C}^n$  and  $v/u$  is holomorphic on  $\mathbb{C}^n$ , then

$$|v(\zeta)/u(\zeta)| \leq C' (1 + \|\zeta\|)^{N'} e^{R'\|Im\zeta\|}$$

for all  $\zeta \in \mathbb{C}^n$ .

The slow decrease condition (ii) implies, in particular, that constant coefficient differential operators on  $\mathbb{R}^n$  are invertible, since their Fourier transforms are polynomials. Below we shall see that it also implies that finite sums of delta functions are invertible. This result is already obtained in [Ehr55], but we include a proof here in order to demonstrate the use of Theorem 2.2(ii) and (2.2).

**Proposition 2.5.** *Fix distinct  $x_1, \dots, x_N$  in  $\mathbb{R}^n$ . Then the distribution  $\mu = \sum_{j=1}^N \delta_{x_j}$  is invertible.*

*Proof.* Without loss of generality, we may assume that  $N > 1$  (otherwise  $c_\mu$  is just a translation which is trivially surjective) and that  $\|x_1\| \geq \|x_j\|$  for all  $j$ . Then in particular,  $x_1 \neq 0$  and  $\langle x_j, x_1 \rangle < \|x_1\|^2$  for all  $j > 1$ . Choose any constant  $A$  such that

$$(2.4) \quad A > \frac{\|x_1\| \log N}{\log 2(\|x_1\|^2 - \langle x_j, x_1 \rangle)} \quad (j = 2, \dots, N).$$

For any  $\zeta \in \mathbb{C}^n$ , we have  $\mu^*(\zeta) = \sum_{j=1}^N e^{-i\langle x_j, \zeta \rangle}$ . Thus if  $\xi, \eta \in \mathbb{R}^n$  and  $\zeta = \xi + i\eta$ , we have

$$(2.5) \quad \begin{aligned} |\mu^*(\zeta)| &= \left| \sum_{j=1}^N e^{\langle x_j, \eta \rangle} e^{-i\langle x_j, \xi \rangle} \right| \\ &= e^{\langle x_1, \eta \rangle} \left| e^{-i\langle x_1, \xi \rangle} + \sum_{j=2}^N e^{(\langle x_j, \eta \rangle - \langle x_1, \eta \rangle)} e^{-i\langle x_j, \xi \rangle} \right| \\ &\geq e^{\langle x_1, \eta \rangle} \left( 1 - \sum_{j=2}^N e^{(\langle x_j, \eta \rangle - \langle x_1, \eta \rangle)} \right) \end{aligned}$$

Fixing  $\xi$  for the moment, let  $\eta = tx_1$ , choosing  $t$  so that

$$\frac{\|x_1\| \log N}{\|x_1\|^2 - \langle x_j, x_1 \rangle} < t\|x_1\| \leq A \log(2 + \|\xi\|) \quad (j = 2, \dots, N).$$

This is possible, by our choice (2.4) of  $A$ . Then  $\|\zeta - \xi\| = \|\eta\| = t\|x_1\| \leq A \log(2 + \|\xi\|)$ , and moreover,  $t(\langle x_j, x_1 \rangle - \|x_1\|^2) < -\log N$  for  $j \geq 2$ . Hence by (2.5) we have

$$\begin{aligned} |\mu^*(\zeta)| &\geq e^{t\|x_1\|^2} \left( 1 - \sum_{j=2}^N e^{t(\langle x_j, x_1 \rangle - \|x_1\|^2)} \right) \\ (2.6) \quad &\geq e^{t\|x_1\|^2} \cdot \frac{1}{N} \end{aligned}$$

If we now choose  $t$  so that  $t\|x_1\| = A \log(2 + \|\xi\|)$ , then we see that the Fourier estimate (2.6) becomes

$$|\mu^*(\zeta)| \geq \frac{1}{N} (2 + \|\xi\|)^{A\|x_1\|},$$

which will certainly imply the slow decrease condition (2.2).  $\square$

*Remark.* For distinct points  $x_1, \dots, x_N$  in  $\mathbb{R}^n$  and nonzero complex scalars  $c_1, \dots, c_N$ , the above proof can be easily modified to show that the distribution

$$\mu = \sum_{j=1}^N c_j \delta_{x_j}$$

is invertible. In this case, we can again assume that  $N > 1$  and that  $\|x_1\|$  is maximal, and we choose  $A$  so that

$$A > \frac{\|x_1\|(\log M + \log |c_j| - \log |c_1|)}{\log 2(\|x_1\|^2 - \langle x_j, x_1 \rangle)} \quad (j = 2, \dots, N),$$

where  $M$  is any constant such that  $M > N$  and  $M > 2|c_j|$  for  $j = 1, \dots, N$ . We will leave the details to the reader.

We finish this section by including the more general result from [Ehr55, Theorem 5] which we will use later

**Theorem 2.6.** *Fix distinct points  $x_1, \dots, x_N$  in  $\mathbb{R}^n$  and let  $p_1, \dots, p_N$  be polynomials in  $\mathbb{R}^n$ . Then the distribution  $\mu = \sum_{j=1}^N p_j(\partial_1, \dots, \partial_n) \delta_{x_j}$  is invertible.*

### 3. NONCOMPACT SYMMETRIC SPACES: PRELIMINARIES AND NOTATION

Now let  $X = G/K$  be a noncompact symmetric space, where  $G$  is a real noncompact semisimple Lie group with finite center, and  $K$  is a maximal compact subgroup. We will now fix the more or less standard terminology associated with these spaces, which may be found, for example, in Helgason's books [Hel01] and [Hel00].

Let  $\mathfrak{g}$  denote the Lie algebra of  $G$  and denote by  $\langle X, Y \rangle$  the Killing form

$$\langle X, Y \rangle = \text{Tr}(\text{ad}(X)\text{ad}(Y)).$$

If  $\mathfrak{k}$  denotes the Lie algebra of  $K$ , then we have a Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where  $\mathfrak{p}$  is the orthogonal complement of  $\mathfrak{k}$  under the Killing form on  $\mathfrak{g}$ . The Killing form restricted to  $\mathfrak{p}$  is positive definite, and we define a norm on  $\mathfrak{p}$  by

$$\|X\| = \sqrt{\langle X, X \rangle}.$$

We endow the symmetric space  $X$  with the left invariant Riemannian metric induced from this norm on  $\mathfrak{p} \cong T_o(G/K)$ , where  $o$  is the identity coset  $\{K\}$  in  $X = G/K$ .

Let  $\mathfrak{a}$  be a maximal abelian subspace of  $\mathfrak{p}$ , and let  $\Sigma$  denote the set of (restricted) roots of  $\mathfrak{g}$  with respect to  $\mathfrak{a}$ . For each  $\alpha \in \Sigma$ , let  $\mathfrak{g}_\alpha$  be the corresponding root space,  $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}$ . Then we have the root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha,$$

with  $\mathfrak{g}_0$  the centralizer of  $\mathfrak{a}$  in  $\mathfrak{g}$ . We have  $\mathfrak{g}_0 = \mathfrak{h}_k \oplus \mathfrak{a}$ , where  $\mathfrak{h}_k$  is the centralizer of  $\mathfrak{a}$  in  $\mathfrak{k}$ .

We fix a positive Weyl chamber  $\mathfrak{a}^+$  in  $\mathfrak{a}$ , and let  $\Sigma^+$  denote the corresponding set of positive restricted roots. In addition, let  $\Sigma_0$  and  $\Sigma_0^+$  denote the set of indivisible roots and positive indivisible roots, respectively. Let

$$\mathfrak{n} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha,$$

and let  $N$  be the analytic subgroup of  $G$  with Lie algebra  $\mathfrak{n}$ . If  $A$  is the analytic subgroup of  $G$  with Lie algebra  $\mathfrak{a}$ , then we have the Iwasawa decomposition

$$(3.1) \quad G = NAK.$$

If  $g \in G$ , we write  $g = n(g) \exp A(g) k(g)$ , in accordance with (3.1), with  $A(g) \in \mathfrak{a}$ .

If  $\alpha \in \Sigma$ , its *multiplicity* is  $m_\alpha = \dim \mathfrak{g}_\alpha$ . We put

$$(3.2) \quad \rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha.$$

Let  $\{\alpha_1, \dots, \alpha_l\}$  be the set of all simple roots. The root lattice  $\Lambda$  is the subset of  $\mathfrak{a}^*$  consisting of all sums  $\sum_{j=1}^l k_j \alpha_j$ , with each  $k_j \in \mathbb{Z}$ . We also put  $\Lambda_+ = \{\sum_{j=1}^l k_j \alpha_j : k_j \in \mathbb{Z}^+ \text{ for all } j\}$ .

For each  $g \in G$ , let  $\tau(g)$  denote the left translation  $x \mapsto g \cdot x$  on  $X$ , and if  $f$  is a function on  $X$ , we put  $\tau(g)f(x) = f(g^{-1} \cdot x)$ .

Let  $M$  be the centralizer and  $M'$  the normalizer of  $A$  in  $K$ , and let  $W$  be the quotient group  $M'/M$ . Then  $W$  is the Weyl group associated with the root system  $\Sigma$ . Let  $\mathbb{D}(X)$  denote the algebra of left invariant differential operators on  $X$ , and let  $\Gamma$  be the Harish-Chandra isomorphism from  $\mathbb{D}(X)$  to the algebra  $I(\mathfrak{a}) = S(\mathfrak{a})^W$  of  $W$ -invariant elements of the symmetric algebra  $S(\mathfrak{a})$ .

A *spherical function* on  $X$  is a  $K$ -invariant joint eigenfunction  $\varphi$  of  $\mathbb{D}(X)$  normalized so that  $\varphi(o) = 1$ . The spherical functions are parametrized by the orbit space  $\mathfrak{a}_\mathbb{C}^*/W$ , where  $\mathfrak{a}_\mathbb{C}^*$  is the complexified dual space of  $\mathfrak{a}$ . The spherical function corresponding to  $\lambda \in \mathfrak{a}_\mathbb{C}^*$  (or rather its  $W$  orbit) is

$$(3.3) \quad \varphi_\lambda(x) = \int_B e^{(i\lambda + \rho)A(x,b)} db \quad (x \in X).$$

Here  $B = K/M$ ,  $db$  is the normalized  $K$ -invariant measure on the coset space  $B$ , and if  $x = g \cdot o$ ,  $b = kM$ , we have put  $A(x, b) = A(k^{-1}g)$ .  $A(x, b)$  represents the “directed distance” from  $o$  to the horocycle passing through  $x$  and with “normal”  $b$ . More precisely, by the Iwasawa decomposition, if  $b = kM$ , then we have  $x = kan \cdot o$  for unique  $a \in A$  (independent of the choice of representative in the coset  $kM$ ) and  $n \in N$ , and  $a = \exp A(x, b)$ .

The spherical function  $\varphi_\lambda$  satisfies

$$(3.4) \quad \varphi_{w \cdot \lambda}(x) = \varphi_\lambda(x), \quad \varphi_\lambda(k \cdot x) = \varphi_\lambda(x)$$

for all  $x \in X$ ,  $k \in K$ ,  $\lambda \in \mathfrak{a}_\mathbb{C}^*$ , and  $w \in W$ .

Later we will need the following integration formula based on the Iwasawa decomposition (3.1). Using the fact that  $A$  normalizes  $N$ , we have  $G = ANK$ , so with appropriate normalizations of the Haar measures on  $A$  and  $N$  we have

$$(3.5) \quad \int_X f(x) dx = \int_A \int_N f(an \cdot o) dn da,$$

when this integral makes sense.

4. THE FOURIER AND RADON TRANSFORMS ON  $G/K$ 

The *Fourier transform* of a function  $f \in \mathcal{D}(X)$  is the function  $\tilde{f}$  on  $\mathfrak{a}_{\mathbb{C}}^* \times B$  given by

$$(4.1) \quad \tilde{f}(\lambda, b) = \int_X f(x) e^{(-i\lambda + \rho) A(x, b)} dx \quad (\lambda \in \mathfrak{a}_{\mathbb{C}}^*, b \in B).$$

The Fourier transform extends naturally to compactly supported distributions on  $X$ : if  $S \in \mathcal{E}'(X)$ ,  $\tilde{S}$  is the function on  $\mathfrak{a}_{\mathbb{C}}^* \times B$  given by

$$\tilde{S}(\lambda, b) = S(e_{-\lambda, b})$$

where  $e_{\lambda, b}(x) = e^{(i\lambda + \rho) A(x, b)}$  which is in  $\mathcal{E}(X)$ . This definition is often written as an integral

$$(4.2) \quad \tilde{S}(\lambda, b) = \int_X e^{(-i\lambda + \rho) A(x, b)} dS(x).$$

In case  $f$  (or  $S$ ) is  $K$ -invariant, the Fourier transform becomes the *spherical Fourier transform*, which for  $S$  is given by

$$(4.3) \quad \tilde{S}(\lambda) = S(\varphi_{-\lambda}) = \int_X \varphi_{-\lambda}(x) dS(x) \quad (\lambda \in \mathfrak{a}_{\mathbb{C}}^*.)$$

Moreover, if  $S, T \in \mathcal{E}'(X)$ , and if  $T$  is  $K$ -invariant, we have

$$(4.4) \quad (S * T)^\sim(\lambda, b) = \tilde{S}(\lambda, b) \tilde{T}(\lambda) \quad (\lambda \in \mathfrak{a}_{\mathbb{C}}^*, b \in K/M),$$

with analogous relations in case  $S$  or  $T$  are replaced by elements of  $\mathcal{D}(X)$ .

The Fourier transform on  $X$  is intimately connected to the *horocycle Radon transform*. A *horocycle* is an orbit in  $X$  of a conjugate of  $N$ . If  $\Xi$  denotes the set of all horocycles, then  $G$  acts transitively on  $\Xi$  and the isotropy subgroup of the “fundamental” horocycle  $\xi_0 = N \cdot o$  is  $MN$ . Thus we can identify  $\Xi$  with the homogeneous manifold  $G/MN$ . Moreover, the Iwasawa decomposition also shows that the map

$$(4.5) \quad \begin{aligned} K/M \times A &\rightarrow \Xi \\ (kM, a) &\mapsto ka \cdot \xi_0 \end{aligned}$$

is a diffeomorphism. The *horocycle Radon transform* maps suitable functions on  $X$  to suitable functions on  $\Xi$ , and is given by

$$(4.6) \quad \hat{f}(ka \cdot \xi_0) = \int_N f(kan \cdot o) dn \quad (k \in K, a \in A).$$

In particular, the map  $f \mapsto \hat{f}$  is a continuous linear map from  $\mathcal{D}(X)$  into  $\mathcal{D}(\Xi)$ . (See [Hel08], Ch. I, §3 for general continuity properties of integral transforms on homogeneous spaces.) The Iwasawa decomposition implies the following “projection-slice” relation between the Fourier and Radon transforms:

$$(4.7) \quad \begin{aligned} \tilde{f}(\lambda, b) &= \int_A \hat{f}(b, a) e^{(-i\lambda + \rho) \log a} da \\ &= (e^{\rho(\log(\cdot))} \hat{f}(b, \cdot))^*(\lambda) \quad ((\lambda, b) \in \mathfrak{a}_{\mathbb{C}}^* \times B). \end{aligned}$$

The *dual horocycle transform*  $\psi \mapsto \check{\psi}$  maps  $\mathcal{E}(\Xi)$  to  $\mathcal{E}(X)$  by integrating over horocycles containing a given point:

$$(4.8) \quad \check{\psi}(g \cdot o) = \int_K \psi(gk \cdot \xi_0) dk \quad (g \in G)$$

for  $\psi \in \mathcal{E}(X)$ . This transform is a continuous map from  $\mathcal{E}(\Xi)$  to  $\mathcal{E}(X)$  (again see [Hel08], Ch. I, §3). Our group-theoretic setup implies that the horocycle transform and its dual are formal adjoints, hence the term *dual transform*:

$$(4.9) \quad \int_{B \times A} \hat{f}(b, a) \psi(b, a) e^{2\rho(\log a)} da db = \int_X f(x) \check{\psi}(x) dx,$$

for  $f \in \mathcal{D}(X)$  and  $\psi \in \mathcal{E}(\Xi)$ . (The  $G$ -invariant measure on  $\Xi = G/MN = K/M \times A$  is  $e^{2\rho(\log a)} da db$ .)

We can use the adjoint relation (4.8) to define the Radon transform of any compactly supported distribution on  $X$ . If  $S \in \mathcal{E}'(X)$ , its Radon transform  $\hat{S}$  is the distribution on  $\Xi$  given by

$$(4.10) \quad \hat{S}(\psi) = S(\check{\psi}) \quad (\psi \in \mathcal{E}(\Xi)).$$

Being the adjoint of the map  $\psi \mapsto \check{\psi}$ , we see that the map  $S \mapsto \hat{S}$  is a continuous linear map from  $\mathcal{E}'(X)$  to  $\mathcal{E}'(\Xi)$ .

To derive a projection-slice theorem for distributions, we first define “restriction” maps  $S \mapsto \hat{S}_b$  from  $\mathcal{E}'(X)$  to  $\mathcal{E}'(A)$  for each  $b \in B$  as follows. Fix  $b \in B$ . If  $F \in \mathcal{E}(A)$ , consider the function  $F^b \in \mathcal{E}(X)$  given by

$$(4.11) \quad F^b(x) = F(\exp A(x, b)) \quad (x \in X).$$

Let  $b = kM$ . Then the function  $F^b$  is constant on horocycles with normal  $b$ ; that is, on the horocycles  $ka \cdot \xi_0$ , and so is a *horocycle plane wave*. Since  $X = kAN \cdot o$ , it is not hard to see that  $F \mapsto F^b$  is a continuous linear map from  $\mathcal{E}(A)$  to  $\mathcal{E}(X)$ .

We define the map  $S \mapsto \hat{S}_b$  to be the adjoint of the map  $F \mapsto F^b$ :

$$(4.12) \quad \hat{S}_b(F) = S(F^b) \quad (F \in \mathcal{E}(A)).$$

The map  $S \mapsto \hat{S}_b$  is therefore a continuous linear map from  $\mathcal{E}'(X)$  to  $\mathcal{E}'(A)$ . In case  $S = f \in \mathcal{D}(X)$ , then by (3.5)

$$\hat{f}_b(a) = \hat{f}(b, a) = \int_N f(kan \cdot o) dn \quad (a \in A, b = kM).$$

From (4.2), we now obtain the projection-slice theorem for distributions:

$$(4.13) \quad \tilde{S}(\lambda, b) = \left( e^\rho \hat{S}_b \right)^* (\lambda)$$

where the Euclidean Fourier transform is taken over  $\mathfrak{a}$ . If  $\mu \in \mathcal{E}'(X)$  is  $K$ -invariant, relation (4.4) shows that

$$(4.14) \quad e^\rho (\widehat{S * \mu})_b = (e^\rho \hat{S}_b) * \mu_A,$$

where the convolution on the right hand side is taken over the Euclidean space  $\mathfrak{a}$ , and  $\mu_A \in \mathcal{E}'(\mathfrak{a})$  is the *Abel transform* of  $\mu$ :

$$(4.15) \quad \mu_A = e^\rho \hat{\mu}_b.$$

Here  $b$  is any element of  $B$ . (Since  $\mu$  is  $K$ -invariant, the choice of  $b$  does not matter.)

Note also that as a special case of (4.13), we have

$$(4.16) \quad (\mu_a)^*(\lambda) = \tilde{\mu}(\lambda) \quad (\lambda \in \mathfrak{a}_\mathbb{C}^*).$$

There are Paley-Wiener theorems that describe the ranges of the Fourier transforms (4.1) and (4.2). To state them properly, we first note that  $\tilde{S}(\lambda, b)$  and  $\tilde{f}(\lambda, b)$  are smooth on  $\mathfrak{a}_\mathbb{C}^* \times B$  and holomorphic in  $\lambda$ ; moreover, the function on  $\mathfrak{a}_\mathbb{C}^*$  given by

$$(4.17) \quad \lambda \mapsto \int_B \tilde{S}(\lambda, b) e^{(i\lambda + \rho)A(x, b)} db$$



turns out to be  $W$ -invariant, and  $\tilde{f}$  satisfies a similar property. The relation (4.17) is a consequence of the functional relation

$$\varphi_\lambda(g^{-1}h \cdot o) = \int_B e^{(-i\lambda+\rho)A(g \cdot o, b)} e^{(i\lambda+\rho)A(h \cdot o, b)} db,$$

for all  $\lambda \in \mathfrak{a}_\mathbb{C}^*$ , and  $g, h \in G$ .

The space  $\mathfrak{a}$  is equipped with the Killing form inner product from  $\mathfrak{p}$ . Fix a basis for  $\mathfrak{a}$  and a dual basis on  $\mathfrak{a}^*$  arising from the Killing form inner product. Denote by  $\langle \lambda, \eta \rangle$  the Euclidean inner product of  $\lambda, \eta \in \mathfrak{a}^*$  in terms of the basis for  $\mathfrak{a}^*$ . Finally extend this inner product to a bilinear form on  $\mathfrak{a}_\mathbb{C}^* = \mathfrak{a}^* + i\mathfrak{a}$ , and let  $\|\lambda\|$  denote the norm of  $\lambda \in \mathfrak{a}_\mathbb{C}^*$  inherited from the bilinear form.

For  $R > 0$ , a function  $\psi(\lambda, b)$ , smooth on  $\mathfrak{a}_\mathbb{C}^* \times B$  and holomorphic in  $\lambda$  is said to be *rapidly decreasing of uniform exponential type  $R$*  provided that for any  $N \in \mathbb{Z}^+$ ,  $\psi$  satisfies the condition

$$(4.18) \quad \sup_{(\lambda, b) \in \mathfrak{a}_\mathbb{C}^* \times B} (1 + \|\lambda\|)^N e^{-R\|\text{Im } \lambda\|} |\psi(\lambda, b)| < \infty.$$

Let  $\mathcal{H}_R(\mathfrak{a}_\mathbb{C}^* \times B)$  denote the vector space of all such functions, and let  $\mathcal{H}(\mathfrak{a}_\mathbb{C}^* \times B)$  be their union for all  $R$ . Let  $\mathcal{H}_R^W(\mathfrak{a}_\mathbb{C}^* \times B)$  denote the subspace of  $\mathcal{H}_R(\mathfrak{a}_\mathbb{C}^* \times B)$  consisting of functions  $\psi$  satisfying the invariance condition (4.17), and let  $\mathcal{H}^W(\mathfrak{a}_\mathbb{C}^* \times B)$  be their union.

Let  $\mathcal{D}_R(X)$  denote the vector space of all  $C^\infty$  functions on  $X$  with support in the closed ball  $\overline{B}_R(o)$ . Then we have the following Paley-Wiener theorem.

**Theorem 4.1.** ([Hel73], Theorem 8.3.) *The Fourier transform  $f \mapsto \tilde{f}$  is a linear bijection from  $\mathcal{D}_R(X)$  onto  $\mathcal{H}_R^W(\mathfrak{a}_\mathbb{C}^* \times B)$ .*

We now state the corresponding Paley-Wiener theorem for compactly supported distributions. For this, we say that a function  $\Psi(\lambda, b)$ , smooth on  $\mathfrak{a}_\mathbb{C}^* \times B$  and holomorphic in  $\lambda$ , is *of uniform exponential type  $R > 0$  in  $\mathfrak{a}_\mathbb{C}^*$  and of slow growth* provided that there exist constants  $A > 0$  and  $N \in \mathbb{Z}^+$  such that

$$(4.19) \quad |\Psi(\lambda, b)| \leq A (1 + \|\lambda\|)^N e^{R\|\text{Im } \lambda\|} \quad ((\lambda, b) \in \mathfrak{a}_\mathbb{C}^* \times B).$$

Let  $\mathcal{K}_R(\mathfrak{a}_\mathbb{C}^* \times B)$  denote the vector space of all such functions, and let  $\mathcal{K}_R^W(\mathfrak{a}_\mathbb{C}^* \times B)$  be the subspace consisting of those functions  $\Psi$  satisfying the  $W$ -invariance condition (4.17). Finally, let  $\mathcal{K}(\mathfrak{a}_\mathbb{C}^* \times B)$  and  $\mathcal{K}^W(\mathfrak{a}_\mathbb{C}^* \times B)$  denote the union of the subspaces  $\mathcal{K}_R$  and  $\mathcal{K}_R^W$ , respectively, for all  $R > 0$ .

Let  $\mathcal{E}'_R(X)$  denote the subspace of  $\mathcal{E}'(X)$  consisting of all distributions with support in  $\overline{B}_R(o)$ .

**Theorem 4.2.** ([Hel08], Ch. III, Corollary 5.9.) *The Fourier transform  $S \mapsto \tilde{S}$  is a linear bijection from  $\mathcal{E}'_R(X)$  onto  $\mathcal{K}_R^W(\mathfrak{a}_\mathbb{C}^* \times B)$ .*

For proofs of the two Paley-Wiener theorems above, see [Hel08], Ch. III, §5.

## 5. A TEMPLATE FOR SURJECTIVITY

Suppose that  $\mu \in \mathcal{E}'(X)$  is  $K$ -invariant. Let  $c_\mu$  be the convolution operator on  $\mathcal{E}(X)$  given by  $c_\mu(f) = f * \mu$ . Any orthonormal basis of  $\mathfrak{a}$  provides a linear isometry from  $\mathfrak{a}_\mathbb{C}^*$  onto  $\mathbb{C}^l$  where  $l = \dim(\mathfrak{a})$ . A function  $u$  on  $\mathfrak{a}_\mathbb{C}^*$  is called *slowly decreasing* if it is slowly decreasing as a function on  $\mathbb{C}^l$ . Our aim in this section is to prove the following theorem.

**Theorem 5.1.** *Let  $\mu$  be a  $K$ -invariant distribution in  $\mathcal{E}'(X)$  whose spherical Fourier transform  $\tilde{\mu}(\lambda)$  is a slowly decreasing function on  $\mathfrak{a}_\mathbb{C}^*$ . Then the convolution operator  $c_\mu: \mathcal{E}(X) \rightarrow \mathcal{E}(X)$  is surjective.*

Note that since  $\mu$  is  $K$ -invariant, the  $W$ -invariance (3.4) of  $\lambda \mapsto \varphi_\lambda(x)$  and (4.3) imply that  $\tilde{\mu}$  is  $W$ -invariant, and the forward Paley-Wiener Theorem (a consequence of the projection-slice theorem (4.13)) shows that  $\tilde{\mu}(\lambda)$  is of exponential type and slow growth in  $\mathfrak{a}_\mathbb{C}^*$ .



*Proof.* Suppose that  $\tilde{\mu}(\lambda)$  is slowly decreasing. We now define the  $K$ -invariant distribution  $\mu^\vee \in \mathcal{E}'(X)$  as follows. Noting that  $\mu$  is determined by its restriction to the (closed) subspace  $\mathcal{E}^\#(X)$  of  $\mathcal{E}(X)$  consisting of all  $K$ -invariant functions, we put

$$\mu^\vee(f) = \int_{G/K} f(g^{-1}K) d\mu(gK) \quad (f \in \mathcal{E}^\#(X)).$$

Note that the function  $gK \mapsto f(g^{-1}K)$  belongs to  $\mathcal{E}^\#(X)$ .

Now the adjoint map to  $c_\mu: \mathcal{E}(X) \rightarrow \mathcal{E}(X)$  is

$$(5.1) \quad \begin{aligned} c_{\mu^\vee}: \mathcal{E}'(X) &\rightarrow \mathcal{E}'(X) \\ T &\mapsto T * \mu^\vee. \end{aligned}$$

We will show that this latter map is injective and has closed range in the strong (and hence weak\*) topology on  $\mathcal{E}'(X)$ . The theorem will then follow from Theorem 2.3. (As mentioned earlier, strongly closed subspaces of  $\mathcal{E}'(X)$  are also weak\* closed.)

The spherical Fourier transform of  $\mu^\vee$  is  $\tilde{\mu}(-\lambda)$ , which is also slowly decreasing, so to simplify the notation, we will replace  $\mu^\vee$  by  $\mu$  and show that the map  $c_\mu: \mathcal{E}'(X) \rightarrow \mathcal{E}'(X)$  is injective and has closed range. Let us first show that the map  $c_\mu$  is injective. The set of all  $\lambda$  for which  $\tilde{\mu}(\lambda) \neq 0$  is open and dense in  $\mathfrak{a}_\mathbb{C}^*$ , and therefore  $T * \mu = 0$  implies that

$$\tilde{T}(\lambda, b) \tilde{\mu}(\lambda) = 0 \quad ((\lambda, b) \in \mathfrak{a}_\mathbb{C}^* \times B),$$

from which we obtain  $\tilde{T}(\lambda, b) \equiv 0$ , and hence  $T = 0$ .

Now we claim that

$$(5.2) \quad \begin{aligned} c_\mu(\mathcal{E}'(X)) \\ = \{T \in \mathcal{E}'(X) : \tilde{T}(\lambda, b)/\tilde{\mu}(\lambda) \text{ is holomorphic in } \lambda \text{ for each } b \in B\} \end{aligned}$$

and that this set is closed in  $\mathcal{E}'(X)$ .

From relation (4.4), it is clear that the left hand side above is contained in the right. On the other hand, suppose that  $T$  belongs to the right hand side above. Since  $\tilde{T}(\lambda, b)/\tilde{\mu}(\lambda)$  is holomorphic for each fixed  $b$ , Proposition 11.1 in Appendix B then implies that  $\tilde{T}(\lambda, b)/\tilde{\mu}(\lambda)$  is smooth on  $\mathfrak{a}_\mathbb{C}^* \times B$ .

By the forward Paley-Wiener Theorem on  $X$ , there exist positive constants  $A$  and  $R$  and an integer  $N \in \mathbb{Z}^+$  (all of which do not depend on  $b$ ) such that  $\tilde{T}$  satisfies the growth condition

$$(5.3) \quad |\tilde{T}(\lambda, b)| \leq A(1 + \|\lambda\|)^N e^{R\|\operatorname{Im} \lambda\|} \quad (\lambda \in \mathfrak{a}_\mathbb{C}^*)$$

for all  $b \in B$ . Hence by Proposition 2.4, there exist positive constants  $A'$  and  $R'$ , and an integer  $N' \in \mathbb{Z}^+$ , such that

$$(5.4) \quad \left| \frac{\tilde{T}(\lambda, b)}{\tilde{\mu}(\lambda)} \right| \leq A'(1 + \|\lambda\|)^{N'} e^{R'\|\operatorname{Im} \lambda\|} \quad (\lambda \in \mathfrak{a}_\mathbb{C}^*)$$

for all  $b \in B$ . It follows that  $\tilde{T}(\lambda, b)/\tilde{\mu}(\lambda) \in \mathcal{K}(\mathfrak{a}_\mathbb{C}^* \times B)$ , and it remains to be shown that it is in  $K^W(\mathfrak{a}_\mathbb{C}^* \times B)$ .

By assumption  $\tilde{T} \in \mathcal{K}^W(\mathfrak{a}_\mathbb{C}^* \times B)$  which means that

$$\int_B \tilde{T}(\lambda, b) e^{(i\lambda + \rho)A(x, b)} db = \int_B \tilde{T}(\sigma\lambda, b) e^{(i\sigma\lambda + \rho)A(x, b)} db.$$

The  $W$ -invariance of  $\tilde{\mu}(\lambda)$  thus gives

$$\int_B \frac{\tilde{T}(\lambda, b)}{\tilde{\mu}(\lambda)} e^{(i\lambda + \rho)A(x, b)} db = \int_B \frac{\tilde{T}(\sigma\lambda, b)}{\tilde{\mu}(\sigma\lambda)} e^{(i\sigma\lambda + \rho)A(x, b)} db$$

for all  $\sigma \in W$  and all  $\lambda \in \mathfrak{a}_\mathbb{C}^*$  for which  $\tilde{\mu}(\lambda) \neq 0$ .

This set is dense in  $\mathfrak{a}_{\mathbb{C}}^*$ , and the relation above therefore holds for all  $\lambda$  by continuity, since the integrands are uniformly continuous on compact sets.

Now that we have established that  $\tilde{T}(\lambda, b)/\tilde{\mu}(\lambda) \in \mathcal{K}^W(\mathfrak{a}_{\mathbb{C}}^* \times B)$ , the Paley-Wiener Theorem on  $X$  (Theorem 4.2) implies that there exists a distribution  $S \in \mathcal{E}'(X)$  such that  $\tilde{S}(\lambda, b) = \tilde{T}(\lambda, b)/\tilde{\mu}(\lambda)$ . Hence  $T = S * \mu$ , proving the range characterization (5.2).

Next let us prove that the right hand side of (5.2) is a closed subset of  $\mathcal{E}'(X)$ . Since the Abel transform  $\mu_a$  satisfies (4.16),  $\mu_a \in \mathcal{E}'(\mathfrak{a})$  is an invertible distribution on the Euclidean space  $\mathfrak{a}$ . Hence by Theorem 2.2, the convolution operator  $v \mapsto v * \mu_a$  on  $\mathcal{E}'(\mathfrak{a})$  has closed range  $c_{\mu_a}(\mathcal{E}'(\mathfrak{a}))$ . Then by relations (5.2), (4.13), and (4.14), we conclude that

$$(5.5) \quad c_{\mu}(\mathcal{E}'(X)) = \{T \in \mathcal{E}'(X) : e^{\rho} \hat{T}_b \in c_{\mu_a}(\mathcal{E}'(\mathfrak{a})) \text{ for all } b \in B\}$$

For each  $b \in B$ , the linear map  $T \mapsto e^{\rho} \hat{T}_b$  from  $\mathcal{E}'(X)$  to  $\mathcal{E}'(\mathfrak{a})$  is continuous. It follows that  $c_{\mu}(\mathcal{E}'(X))$  is a closed subspace of  $\mathcal{E}'(X)$ , since  $\mathcal{E}'(\mathfrak{a}) * \mu_a$  is closed in  $\mathcal{E}'(\mathfrak{a})$ . This also of course proves that  $c_{\mu^{\vee}}(\mathcal{E}'(X))$  is closed in  $\mathcal{E}'(X)$ , finishing the proof of Theorem 5.1.  $\square$

**Corollary 5.2.** (*Helgason, 1973*) *Every nonzero  $G$ -invariant differential operator on  $X$  is a surjective map from  $\mathcal{E}(X)$  onto  $\mathcal{E}(X)$ .*

This is one of the main results in [Hel73]. Note that if  $D \in \mathcal{D}(X)$ , then  $Df = f * D\delta_o$ . Now  $D\delta_o \in \mathcal{E}'(X)$  is  $K$ -invariant, and

$$(D\delta_o)^{\sim}(\lambda) = \Gamma(D)(i\lambda).$$

Since the right hand side is a polynomial in  $\lambda$ , it is slowly decreasing, so Theorem 5.1 applies.

## 6. MEAN VALUE OPERATORS ON SYMMETRIC SPACES

Fix a point  $y \in X$ . The *mean value operator*  $M^y$  is defined on suitable functions  $f$  on  $X$  by

$$(6.1) \quad M^y f(x) = \int_K f(gk \cdot y) dk \quad (x = g \cdot o \in X),$$

where  $dk$  is the normalized Haar measure on  $K$ . If  $X$  is of rank one, then the translated orbit  $gK \cdot y$  is the sphere in  $X$  of radius  $d(o, y)$  (where  $d$  denotes the distance in  $X$ ) and center  $g \cdot o$ , so the integral in (6.1) represents the average value of  $f$  on this sphere.

Now choose any  $g_0 \in G$  such that  $y = g_0 \cdot o$ . Then in terms of the convolution on  $X$ , we have

$$(6.2) \quad M^y f = f * \chi_{K \cdot g_0^{-1} \cdot o} \quad (f \in \mathcal{E}(X))$$

where  $\chi_{K \cdot g_0^{-1} \cdot o} \in \mathcal{E}'(X)$  is the distribution on  $X$  given by

$$\varphi \mapsto \int_K \varphi(kg_0^{-1} \cdot o) dk \quad (\varphi \in \mathcal{E}(X)).$$

This distribution is  $K$ -invariant and is clearly independent of the choice of  $g_0$ .

Note that for  $h \in A$  we have

$$(6.3) \quad (\chi_{K \cdot h^{-1} \cdot o})^{\sim}(\lambda) = \varphi_{\lambda}(h) \quad (\lambda \in \mathfrak{a}_{\mathbb{C}}^*).$$

Therefore by Theorem 5.1, we see that

**Proposition 6.1.** *Let  $h \in A$  be fixed. If the function  $\lambda \mapsto \varphi_{\lambda}(h)$  is slowly decreasing on  $\mathfrak{a}_{\mathbb{C}}^*$ , then  $M^h : \mathcal{E}(X) \rightarrow \mathcal{E}(X)$  is surjective.*

7. THE CASE OF COMPLEX  $G$ 

When  $G$  is complex, then  $K$  is a compact real form of  $G$  and  $\mathfrak{h} = \mathfrak{a} + i\mathfrak{a}$  is a Cartan subalgebra of  $\mathfrak{g}$ . Let  $\Delta$  be the set of roots of  $\mathfrak{g}$  with respect to  $\mathfrak{h}$ , let  $\Delta^+$  be a fixed choice of positive roots, and let  $\rho = \sum_{\alpha \in \Delta^+} \alpha$ . Let  $W$  be the Weyl group corresponding to  $\Delta$ .

Then for any  $h \in A$ , we have

$$(7.1) \quad \varphi_\lambda(h) = c \frac{\pi(\rho)}{\pi(i\lambda)} \frac{\sum_{s \in W} \det(s) e^{is\lambda(H)}}{\sum_{s \in W} \det(s) e^{s\rho(H)}},$$

where  $\pi(\lambda) = \prod_{\alpha \in \Delta^+} \alpha(\lambda)$ . (See [Hel00], Ch. IV, Theorem 5.7.)

Our objective is to show that, for fixed  $h \in A$ , the holomorphic function  $\lambda \mapsto \varphi_\lambda(h)$  is slowly decreasing on  $\mathfrak{a}_\mathbb{C}^*$ .

Now  $h = \exp H$  for a unique  $H \in \mathfrak{a}$ . We first consider the case when  $H \in \mathfrak{a}$  is *regular*; that is to say,  $\alpha(H) \neq 0$  for all  $\alpha \in \Sigma$ . Then  $sH \neq s'H$  for all  $s \neq s'$  in  $W$ , and the denominator in (7.1) does not vanish. By the remark after Proposition 2.5, the distribution on  $\mathfrak{a}$  given by  $T = \sum_{s \in W} (\det s) \delta_{-sH}$  is invertible, so its Fourier transform

$$T^*(\lambda) = \sum_{s \in W} \det s e^{is\lambda(H)}$$

is slowly decreasing. It is also divisible by the polynomial  $\pi(\lambda)$  in the algebra  $\mathcal{H}(\mathfrak{a}_\mathbb{C}^*)$ . Hence by Condition (iii) of Theorem 2.2, the function

$$\varphi_\lambda(\exp H) = \frac{\pi(\rho)}{\sum_{s \in W} (\det s) e^{s\rho(H)}} \cdot \frac{T^*(\lambda)}{\pi(\lambda)}$$

is slowly decreasing. Theorem 5.1 now implies that if  $\mu$  is the  $K$ -invariant distribution  $\chi_{K \cdot h^{-1} \cdot o}$  on  $X$ , then  $c_\mu : \mathcal{E}(X) \rightarrow \mathcal{E}(X)$  is surjective.

Suppose now that  $H$  is not regular. The function  $\lambda \mapsto \varphi_\lambda(h)$  on  $\mathfrak{a}_\mathbb{C}^*$  is of course still holomorphic of exponential type, but the formula (7.1) for  $\varphi_\lambda(h)$  needs to be modified since in the present case the “Weyl denominator”  $\sum_{s \in W} \det(s) e^{s\rho(H)}$  equals 0.

This Weyl denominator can also be written

$$(7.2) \quad \sum_{s \in W} \det(s) e^{s\rho(H)} = \prod_{\alpha \in \Delta^+} (e^{\alpha(H)} - e^{-\alpha(H)}).$$

(See Lemma 24.3 in [Hum78].) Now let  $\Delta_0$  denote the root system  $\{\alpha \in \Delta \mid \alpha(H) = 0\}$ , and let  $\Delta_0^+ = \Delta_0 \cap \Delta^+$ . The Weyl group of  $\Delta_0$  is the subgroup  $W_0$  of  $W$  consisting of all elements which leave  $H$  fixed, and is generated by the reflections along the root hyperplanes  $\pi_\alpha = \alpha^\perp$ , where  $\alpha \in \Delta_0$  (or even the *simple* root hyperplanes in  $\Delta_0$ .) Let  $\rho_0 = (1/2) \sum_{\alpha \in \Delta_0^+} \alpha$ . The Weyl denominator corresponding to  $\Delta_0$  is

$$(7.3) \quad \sum_{s \in W_0} \det(s) e^{s\rho_0(H)} = \prod_{\alpha \in \Delta_0^+} (e^{\alpha(H)} - e^{-\alpha(H)}).$$

Let  $\pi_0$  denote the polynomial on  $\mathfrak{a}_\mathbb{C}^*$  given by  $\pi_0(\lambda) = \prod_{\alpha \in \Delta_0^+} \alpha(\lambda)$  and let  $|W_0|$  denote the order of  $W_0$ . To obtain an explicit expression for  $\varphi_\lambda(\exp H)$ , we first calculate  $\varphi_\lambda(\exp(H + tH_{\rho_0}))$ , and note that for small positive  $t$ , the vector  $H + tH_{\rho_0}$  is regular. Then the fraction on the right hand side of (7.1) is

$$\begin{aligned} & \frac{\sum_{s \in W} \det(s) e^{is\lambda(H + tH_{\rho_0})}}{\prod_{\alpha \in \Delta^+} (e^{\alpha(H + tH_{\rho_0})} - e^{-\alpha(H + tH_{\rho_0})})} \\ &= \frac{1}{|W_0|} \frac{\sum_{\sigma \in W_0} \sum_{s \in W} \det(\sigma^{-1}s) e^{i\sigma^{-1}s\lambda(H + tH_{\rho_0})}}{\prod_{\alpha \in \Delta^+} (e^{\alpha(H + tH_{\rho_0})} - e^{-\alpha(H + tH_{\rho_0})})} \end{aligned}$$

The right hand side above can be written

$$\frac{1}{|W_0|} \frac{\sum_{s \in W} \det(s) e^{is\lambda(H)} \sum_{\sigma \in W_0} \det(\sigma) e^{is\lambda(t\sigma H_{\rho_0})}}{\prod_{\alpha \in \Delta^+ \setminus \Delta_0^+} (e^{\alpha(H+tH_{\rho_0})} - e^{-\alpha(H+tH_{\rho_0})}) \prod_{\alpha \in \Delta_0^+} (e^{\alpha(tH_{\rho_0})} - e^{-\alpha(tH_{\rho_0})})},$$

which by (7.3) equals

$$\frac{1}{|W_0|} \frac{\sum_{s \in W} \det(s) e^{is\lambda(H)} \prod_{\alpha \in \Delta_0^+} (e^{\alpha(it s \lambda)} - e^{-\alpha(it s \lambda)})}{\prod_{\alpha \in \Delta^+ \setminus \Delta_0^+} (e^{\alpha(H+tH_{\rho_0})} - e^{-\alpha(H+tH_{\rho_0})}) \prod_{\alpha \in \Delta_0^+} (e^{\alpha(tH_{\rho_0})} - e^{-\alpha(tH_{\rho_0})})},$$

Taking the limit as  $t \rightarrow 0$ , we obtain

$$(7.4) \quad \varphi_\lambda(\exp H) = \frac{\pi(\rho)}{|W_0| \pi(i\lambda) \pi_0(\rho_0)} \frac{\sum_{s \in W} \det(s) \pi_0(is\lambda) e^{is\lambda(H)}}{\prod_{\alpha \in \Delta^+ \setminus \Delta_0^+} (e^{\alpha(H)} - e^{-\alpha(H)})}$$

The exponential polynomial  $\psi(\lambda = \sum_{s \in W} \det(s) \pi_0(is\lambda) e^{is\lambda(H)})$  is skew in  $\lambda$ , meaning that  $\psi(\sigma\lambda) = \det \sigma \psi(\lambda)$  for all  $\sigma \in W$ . This makes  $\psi(\lambda)$  divisible by  $\pi(i\lambda)$ , so the right hand side of (7.4) represents a holomorphic function of  $\lambda$ .

**Theorem 7.1.** *Let  $X = G/K$ , with  $G$  complex. For any  $h \in A$ , the mean value operator*

$$M^h: \mathcal{E}(X) \rightarrow \mathcal{E}(X)$$

*is surjective.*

## 8. THE CASE OF A GENERAL NONCOMPACT SYMMETRIC SPACE

We return to the case of a general noncompact symmetric space  $X = G/K$ . For  $M \geq 0$  let  $\mathfrak{a}_M^+$  be the subchamber  $\{H \in \mathfrak{a} : \alpha(H) > M \text{ for all } \alpha \in \Sigma^+\}$ , and let  $A_M^+ = \exp \mathfrak{a}_M^+$ . Our aim in this subsection is to prove the following result.

**Theorem 8.1.** *There exists a constant  $M > 0$  such that the mean value operator  $M^h: \mathcal{E}(X) \rightarrow \mathcal{E}(X)$  is surjective for all  $h \in A_M^+$ .*

We believe that the theorem will be true for all  $h$  in  $A$ , but we are presently not aware of a proof.

By Theorem 5.1 we will need to prove that a constant  $M$  can be found such that for any  $h \in A_M^+$  the holomorphic function  $\lambda \mapsto \varphi_\lambda(h)$  on  $\mathfrak{a}_\mathbb{C}^*$  is slowly decreasing. The key tool is Harish-Chandra's spherical function expansion

$$(8.1) \quad \varphi_\lambda(\exp H) = \sum_{s \in W} c(s\lambda) \Phi_{s\lambda}(H),$$

where  $\Phi_\lambda(H)$  is the *Harish-Chandra series*

$$(8.2) \quad \Phi_\lambda(H) = \sum_{\mu \in \Lambda} \Gamma_\mu(\lambda) e^{(is\lambda - \rho - \mu)(H)}.$$

(See, for example, [Hel00], Chapter IV, for a derivation and treatment.) In (8.2) the coefficients  $\Gamma_\mu(\lambda)$ ,  $\mu \in \Lambda$  are defined by the recursion formula

$$(8.3) \quad \begin{aligned} \Gamma_0(\lambda) &= 1 \\ (\langle \mu, \mu \rangle - 2i \langle \mu, \lambda \rangle) \Gamma_\mu(\lambda) \\ &= 2 \sum_{\alpha \in \Sigma^+} m_\alpha \sum_{\substack{k > 1, \\ \mu - 2k\alpha \in \Lambda^+}} (\langle \mu + \rho - 2k\alpha, \alpha \rangle - i \langle \lambda, \alpha \rangle) \Gamma_{\mu - 2k\alpha}(\lambda) \end{aligned}$$

and in the expansion (8.1),  $c(\lambda)$  is Harish-Chandra's  $c$  function

$$(8.4) \quad c(\lambda) = c_0 \prod_{\alpha \in \Sigma_0^+} \frac{2^{-i\langle \lambda, \alpha_0 \rangle} \Gamma(i\langle \lambda, \alpha_0 \rangle)}{\Gamma\left(\frac{1}{2}\left(\frac{m_\alpha}{2} + m_{2\alpha} + \langle i\lambda, \alpha_0 \rangle\right)\right) \Gamma\left(\frac{1}{2}\left(\frac{m_\alpha}{2} + 1 + \langle i\lambda, \alpha_0 \rangle\right)\right)},$$

with  $\alpha_0 = \alpha / \langle \alpha, \alpha \rangle$  and

$$c_0 = \Gamma((1/2)(m_\alpha + m_{2\alpha} + 1)) 2^{(1/2)m_\alpha + m_{2\alpha}}.$$

The  $c$  function is a meromorphic function on  $\mathfrak{a}_\mathbb{C}^*$  with poles in the hyperplanes  $i\langle \lambda, \alpha_0 \rangle = -m$ , for all  $\alpha \in \Sigma_0^+$  and  $m \in \mathbb{Z}^+$ .

The equality (8.2) holds (and the series  $\Phi_{s\lambda}(H)$  converge for all  $s \in W$ ) when  $H \in \mathfrak{a}^+$  and  $\lambda \in \mathfrak{a}_\mathbb{C}^*$  satisfy  $\langle \mu, \mu \rangle - 2i\langle \mu, s\lambda \rangle \neq 0$  for all  $\mu \in \Lambda \setminus \{0\}$  and  $i(s\lambda - s'\lambda) \notin \tilde{\Lambda}$  for all  $s \neq s'$  in  $W$ .

In particular, the Harish-Chandra series (8.2) converges for all  $\lambda \in \mathfrak{a}^*$ . While the expansion (8.2) is employed mostly to study the spherical function  $\varphi_\lambda(\exp H)$  as a function of  $H$  (for fixed  $\lambda$ ), for the purpose of proving Theorem 8.1 we would like to examine its behavior as  $\lambda$  varies, with  $H$  fixed, while maintaining the prescribed limitations on  $\lambda$ .

Explicitly, in view of the slow decrease criterion (2.2) we would like to use the Harish-Chandra expansion to show that there exists a constant  $M \geq 0$  such that for all  $H \in \mathfrak{a}_M^+$ , there are positive constants  $A, B, C$ , and  $D$  (depending on  $H$ ) for which

$$(8.5) \quad \sup\{|\varphi_\lambda(\exp H)| : \lambda \in \mathfrak{a}_\mathbb{C}^*, \|\lambda - \xi\| \leq A \log(2 + \|\xi\|)\} \geq B(C + \|\xi\|)^{-D}$$

for all  $\xi \in \mathfrak{a}^*$ .

We start with a technical lemma which gives an estimate for the coefficients  $\Gamma_\mu(\lambda)$  when the imaginary part of  $\lambda$  is bounded by a given fixed constant.

**Lemma 8.2.** *Suppose that  $\eta \in \mathfrak{a}^*$  satisfies  $\|\eta\| < (1/4)\|\mu\|$  for all  $\mu \in \Lambda \setminus \{0\}$ . For any vector  $H_0 \in \mathfrak{a}^+$ , there is a constant  $K_{H_0}$  such that*

$$(8.6) \quad |\Gamma_\mu(\xi + i\eta)| \leq K_{H_0} e^{\mu(H_0)}$$

for all  $\mu \in \Lambda$  and all  $\xi \in \mathfrak{a}^*$ .

Since  $\Lambda \setminus \{0\}$  has no accumulation point, the set of all such  $\eta$  is a nonempty open ball in  $\mathfrak{a}^*$ .

*Proof.* For any  $\xi \in \mathfrak{a}^*$  and any  $\mu \in \Lambda \setminus \{0\}$ , our condition for  $\eta$  implies that

$$(8.7) \quad \begin{aligned} |\langle \mu, \mu \rangle - 2i\langle \mu, \xi + i\eta \rangle| &= |\langle \mu, \mu \rangle + 2\langle \mu, \eta \rangle - 2i\langle \mu, \xi \rangle| \\ &\geq \langle \mu, \mu \rangle - 2\|\mu\| \|\eta\| \\ &> \frac{1}{2}\langle \mu, \mu \rangle. \end{aligned}$$

Hence  $\Gamma_\mu(\xi + i\eta)$  is well-defined for each  $\mu \in \Lambda$ .

Now recall that in [Gan71] the *radial density function* on  $\mathfrak{a}^+$  is given by

$$\delta(H) = \prod_{\alpha \in \Sigma^+} (e^{\alpha(H)} - e^{-\alpha(H)})^{m_\alpha}$$

We then have series expansions

$$\begin{aligned}\delta^{1/2}(H) &= e^{\rho(H)} \sum_{\nu \in \Lambda} b_\nu e^{-\nu(H)}, \\ \delta^{-1/2}(H) &= e^{-\rho(H)} \sum_{\nu \in \Lambda} c_\nu e^{-\nu(H)}, \\ \delta^{-1/2}(H) L_{\mathfrak{a}}(\delta^{1/2})(H) &= \sum_{\nu \in \Lambda} d_\nu e^{-\nu(H)},\end{aligned}$$

for  $H \in \mathfrak{a}^+$ , where the coefficients  $b_\nu$ ,  $c_\nu$ , and  $d_\nu$  all grow polynomially in  $\|\nu\|$ , and  $d_0 = \langle \rho, \rho \rangle$ .

Gangolli's modification of the Harish-Chandra series is given by

$$\begin{aligned}\Psi_\lambda(H) &= \delta^{1/2}(H) \Phi_\lambda(H) \\ (8.8) \quad &= \sum_{\mu \in \Lambda} A_\mu(\lambda) e^{(i\lambda - \mu)(H)}\end{aligned}$$

The coefficients  $A_\mu(\lambda)$  satisfy the recurrence relation

$$(8.9) \quad (\langle \mu, \mu \rangle - 2i\langle \mu, \lambda \rangle) A_\mu(\lambda) = \sum_{\substack{\nu \in \Lambda, \nu > 0 \\ \mu - \nu \in \Lambda}} A_{\mu - \nu}(\lambda) d_\nu$$

Now in Gangolli's paper [Gan71], the inequality (8.7) (without the factor  $1/2$ ) was used for  $\lambda \in \mathfrak{a}^* + i\mathfrak{a}_+^*$  to prove that there exists a constant  $C_{H_0}$  such that  $|A_\mu(\lambda)| \leq C_H e^{\mu(H_0)}$  for all  $\mu \in \Lambda$ ; the relation

$$\Gamma_\mu(\lambda) = \sum_{\substack{\nu \in \Lambda \\ \mu - \nu \in \Lambda}} c_\nu A_{\mu - \nu}(\lambda)$$

then implies that there exists a constant  $D_{H_0}$  such that

$$|\Gamma_\mu(\lambda)| \leq D_{H_0} e^{\mu(H_0)}$$

for all  $\lambda \in \mathfrak{a}^* + i\mathfrak{a}_+^*$  and all  $\mu \in \Lambda$ .

Because the inequality (8.7) also holds for  $\lambda = \xi + i\eta$ , the very same proof shows that there is a constant  $K_{H_0}$  satisfying the inequality (8.6). This finishes the proof of the lemma.  $\square$

Our aim is to find a lower bound for the supremum in (8.5). This estimate is accomplished using  $\lambda = \xi - i\eta$  for a fixed  $\eta \in \mathfrak{a}_+^*$ . This restricts the range of the parameter  $\lambda$ , but it is sufficient for our purposes. From now on we will fix an element  $\eta \in \mathfrak{a}^*$  satisfying the five conditions below. Condition (b) is assumed to hold for all  $\mu \in \Lambda \setminus \{0\}$ , and Conditions (c)–(e) are assumed to hold for all  $s \in W$ , and all  $\alpha \in \Sigma_0^+$ :

- (a)  $\eta \in \mathfrak{a}_+^*$ ;
- (b)  $\|\eta\| < (1/4) \|\mu\|$ ;
- (c)  $\langle s\eta, \alpha_0 \rangle \notin -\mathbb{Z}^+$ ;
- (d)  $\langle s\eta, \alpha_0 \rangle + m_\alpha/2 + m_{2\alpha} \notin -2\mathbb{Z}^+$ ;
- (e)  $\langle s\eta, \alpha_0 \rangle + m_\alpha/2 + 1 \notin -2\mathbb{Z}^+$ .

Since  $\mathfrak{a}_+^*$  is an open cone in  $\mathfrak{a}^*$  with vertex at 0, the set of all  $\eta$  satisfying (a) and (b) is a nonempty open subset of  $\mathfrak{a}^*$ . Conditions (c)–(e) are needed in order to apply Stirling's formula for the Gamma function, and they stipulate that  $\eta$  does not belong to a countable set of hyperplanes in  $\mathfrak{a}^*$ . Their union is a set of measure zero in  $\mathfrak{a}^*$ , so there will be elements  $\eta \in \mathfrak{a}^*$  satisfying (a)–(e).

**Lemma 8.3.** *Let  $\eta$  be the fixed element of  $\mathfrak{a}^*$  chosen above, and let  $s \in W$ . Then*

$$(8.10) \quad |c(s(\xi - i\eta))| \asymp \prod_{\alpha \in \Sigma_0^+} (1 + |\langle \xi, \alpha_0 \rangle|)^{-(m_\alpha + m_{2\alpha})/2}$$

for all  $\xi \in \mathfrak{a}^*$ .

The symbol  $\asymp$  in (8.10) means that there are positive constants  $r_1$  and  $r_2$  (which depend on  $s$ ) such that

$$(8.11) \quad r_1 \prod_{\alpha \in \Sigma_0^+} (1 + |\langle \xi, \alpha_0 \rangle|)^{-(m_\alpha + m_{2\alpha})/2} \leq |c(s(\xi - i\eta))| \leq r_2 \prod_{\alpha \in \Sigma_0^+} (1 + |\langle \xi, \alpha_0 \rangle|)^{-(m_\alpha + m_{2\alpha})/2}$$

for all  $\xi \in \mathfrak{a}^*$ .

*Proof.* Using the identity  $\Gamma(2z) = \pi^{-1/2} 2^{2z-1} \Gamma(z) \Gamma(z + 1/2)$  the  $c$  function formula (8.4) becomes

$$c(\lambda) = c_1 \prod_{\alpha \in \Sigma_0^+} \frac{\Gamma\left(\frac{i\langle \lambda, \alpha_0 \rangle}{2}\right) \Gamma\left(\frac{i\langle \lambda, \alpha_0 \rangle}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{m_\alpha}{4} + \frac{m_{2\alpha}}{2} + \frac{\langle i\lambda, \alpha_0 \rangle}{2}\right) \Gamma\left(\frac{m_\alpha}{4} + \frac{1}{2} + \frac{\langle i\lambda, \alpha_0 \rangle}{2}\right)}$$

for some positive constant  $c_1$ . Then putting  $\lambda = \xi - i\eta$ , we obtain

$$(8.12) \quad c(s(\xi - i\eta)) = c_1 \prod_{\alpha \in \Sigma_0^+} \frac{\Gamma\left(\frac{\langle s\eta, \alpha_0 \rangle + i\langle s\xi, \alpha_0 \rangle}{2}\right) \Gamma\left(\frac{\langle s\eta, \alpha_0 \rangle + i\langle s\xi, \alpha_0 \rangle}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{m_\alpha}{4} + \frac{m_{2\alpha}}{2} + \frac{\langle s\eta, \alpha_0 \rangle + i\langle s\xi, \alpha_0 \rangle}{2}\right) \Gamma\left(\frac{m_\alpha}{4} + \frac{1}{2} + \frac{\langle s\eta, \alpha_0 \rangle + i\langle s\xi, \alpha_0 \rangle}{2}\right)}$$

Conditions ((c)–(e) for  $\eta$  ensure that all the Gamma functions on the right hand side of (8.12) are well-defined for all  $\xi \in \mathfrak{a}^*$  and all  $s \in W$ . Since the Gamma function has no zeros, for our fixed  $\eta$  we also see that the right hand side above never vanishes for all  $s \in W$  and all  $\xi \in \mathfrak{a}^*$ .

Now we use the following asymptotic formula for the ratio of two Gamma functions

$$(8.13) \quad \frac{\Gamma(z)}{\Gamma(z+b)} \approx z^{-b} (1 + O(1/z)) \text{ as } |z| \rightarrow \infty,$$

for  $b > 0$ , which is valid as long as  $-\pi + \delta < \text{Arg } z < \pi - \delta$ , for small positive  $\delta$ . (See, for example, Formula 6.1.47 in [AS64].)

For each  $\xi \in \mathfrak{a}^*$ , the real part of the argument in each of the Gamma functions on the right hand side of (8.12) is constant (and not an integer  $\leq 0$ ), so in particular the asymptotic formula (8.13) implies that

$$\left| \frac{\Gamma\left(\frac{\langle s\eta, \alpha_0 \rangle + i\langle s\xi, \alpha_0 \rangle}{2}\right)}{\Gamma\left(\frac{m_\alpha}{4} + \frac{m_{2\alpha}}{2} + \frac{\langle s\eta, \alpha_0 \rangle + i\langle s\xi, \alpha_0 \rangle}{2}\right)} \right| \asymp (1 + |\langle s\xi, \alpha_0 \rangle|)^{-\frac{m_\alpha}{4} + \frac{m_{2\alpha}}{2}}$$

and

$$\left| \frac{\Gamma\left(\frac{\langle s\eta, \alpha_0 \rangle + i\langle s\xi, \alpha_0 \rangle}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{m_\alpha}{4} + \frac{1}{2} + \frac{\langle s\eta, \alpha_0 \rangle + i\langle s\xi, \alpha_0 \rangle}{2}\right)} \right| \asymp (1 + |\langle s\xi, \alpha_0 \rangle|)^{-\frac{m_\alpha}{4}}$$

for any  $\alpha \in \Sigma_0^+$ . Hence (8.12) implies that

$$\begin{aligned} |c(s(\xi - i\eta))| &\asymp \prod_{\alpha \in \Sigma_0^+} (1 + |\langle \xi, s^{-1}\alpha_0 \rangle|)^{-\frac{m_\alpha + m_{2\alpha}}{2}} \\ &= \prod_{\alpha \in \Sigma_0^+} (1 + |\langle \xi, \alpha_0 \rangle|)^{-\frac{m_\alpha + m_{2\alpha}}{2}}, \end{aligned}$$

proving the asymptotic relation (8.10). □



Let us now prove Theorem 8.1. Again we recall that we have fixed the element  $\eta \in \mathfrak{a}^*$  satisfying Conditions (a)–(e) above. According to Lemma 8.3, there are constants  $m_1$  and  $m_2$  such that

$$|c(\xi - i\eta)| \geq m_1 \prod_{\alpha \in \Sigma_0^+} (1 + |\langle \xi, \alpha_0 \rangle|)^{-\frac{m_\alpha + m_{2\alpha}}{2}}$$

for all  $\xi \in \mathfrak{a}^*$ , and

$$|c(s(\xi - i\eta))| \leq m_2 \prod_{\alpha \in \Sigma_0^+} (1 + |\langle \xi, \alpha_0 \rangle|)^{-\frac{m_\alpha + m_{2\alpha}}{2}}$$

for all  $\xi \in \mathfrak{a}^*$  and all  $s$  in  $W$ .

For the moment let us fix a vector  $H_0 \in \mathfrak{a}^+$ . Since  $\eta \in \mathfrak{a}^*$  satisfies Condition (a), Lemma 8.2 implies that there is a positive constant  $K_{H_0}$  for which

$$|\Gamma_\mu(s(\xi - i\eta))| \leq K_{H_0} e^{\mu(H_0)}$$

for all  $\mu \in \Lambda$ , all  $\xi \in \mathfrak{a}^*$ , and all  $s \in W$ . Let  $M_1$  be any number larger than  $|\alpha(H_0)|$  for all  $\alpha \in \Sigma_0^+$ . Then by (8.2), (8.10), and the estimate above, the sum (8.1) representing  $\varphi_{\xi-i\eta}(H)$  converges uniformly on the region  $(\xi, H) \in \mathfrak{a}^* \times \mathfrak{a}_{M_1}$ .

In particular, for  $(\xi, H) \in \mathfrak{a}^* \times \mathfrak{a}_{M_1}$ , we have

$$\begin{aligned} & e^{\rho(H)} |\varphi_{\xi-i\eta}(\exp H)| \\ &= \left| \sum_{s \in W} c(s(\xi - i\eta)) \sum_{\mu \in \Lambda} \Gamma_\mu(s(\xi - i\eta)) e^{s\eta(H) + is\xi(H) - \mu(H)} \right| \\ &\geq |c(\xi - i\eta)| e^{\eta(H)} - |c(\xi - i\eta)| \sum_{\mu \in \Lambda \setminus \{0\}} |\Gamma_\mu(\xi - i\eta)| e^{\eta(H) - \mu(H)} \\ &\quad - \sum_{s \neq e} |c(s(\xi - i\eta))| \sum_{\mu \in \Lambda} |\Gamma_\mu(s(\xi - i\eta))| e^{s\eta(H) - \mu(H)} \\ (8.14) \quad &\geq e^{\eta(H)} \prod_{\alpha \in \Sigma_0^+} (1 + |\langle \xi, \alpha_0 \rangle|)^{-\frac{m_\alpha + m_{2\alpha}}{2}} \times \\ &\quad \left( m_1 - m_2 \sum_{\mu \in \Lambda \setminus \{0\}} K_{H_0} e^{\mu(H_0) - \mu(H)} - m_2 \sum_{s \neq e} e^{s\eta(H) - \eta(H)} \sum_{\mu \in \Lambda} K_{H_0} e^{\mu(H_0) - \mu(H)} \right) \end{aligned}$$

Let  $\alpha_1, \dots, \alpha_l$  be the simple roots in  $\mathfrak{a}_+^*$ , and let  $H_1, \dots, H_l$  be a dual basis of  $\mathfrak{a}$ . If  $H = \sum_{j=1}^l k_j H_j \in \mathfrak{a}$ , then  $H \in \mathfrak{a}^+$  if and only if each  $k_j > 0$  and for any  $M > 0$ ,  $H \in \mathfrak{a}_M$  if and only if each  $k_j > M$ .

Let  ${}^+\mathfrak{a}^*$  be the dual cone  $\{\lambda \in \mathfrak{a}^* : \lambda(H) > 0 \text{ for all } H \in \mathfrak{a}^+\}$ . Then  $\lambda \in {}^+\mathfrak{a}^*$  if and only if  $\lambda$  is nonzero and  $\lambda = \sum_{j=1}^l m_j \alpha_j$ , where each  $m_j \geq 0$ . For  $\lambda \in {}^+\mathfrak{a}^*$  we put  $m(\lambda) = \sum_{j=1}^l m_j$ . Since  $\eta \in \mathfrak{a}_+^*$ , we have  $\eta - s\eta \in {}^+\mathfrak{a}^*$  for all  $s \neq e$  in  $W$ . (See, for instance, [Hel01], Ch. VII, Theorem 2.12.)

Let  $M > M_1$ . If  $H \in \mathfrak{a}_M^+$ , then  $\mu(H) > M m(\mu)$  and  $\eta(H) - s\eta(H) > M m(\eta - s\eta)$ . Thus the relation (8.14) implies that

$$\begin{aligned} (8.15) \quad & e^{\rho(H)} |\varphi_{\xi-i\eta}(\exp H)| \geq e^{\eta(H)} \prod_{\alpha \in \Sigma_0^+} (1 + |\langle \xi, \alpha_0 \rangle|)^{-\frac{m_\alpha + m_{2\alpha}}{2}} \times \\ & \left( m_1 - m_2 K_{H_0} e^{\mu(H_0)} \left( \sum_{\mu \in \Lambda \setminus \{0\}} e^{-M m(\mu)} + \sum_{s \neq e} e^{-M m(\eta - s\eta)} \sum_{\mu \in \Lambda} e^{-M m(\mu)} \right) \right) \end{aligned}$$

Since both  $\sum_{\mu \in \Lambda \setminus \{0\}} e^{-M m(\mu)}$  and  $\sum_{s \neq e} e^{-M m(\eta - s\eta)}$  tend to 0 as  $M \rightarrow \infty$ , the expression

$$(8.16) \quad m_1 - m_2 K_{H_0} e^{\mu(H_0)} \left( \sum_{\mu \in \Lambda \setminus \{0\}} e^{-M m(\mu)} + \sum_{s \neq e} e^{-M m(\eta - s\eta)} \sum_{\mu \in \Lambda} e^{-M m(\mu)} \right)$$

is positive for all sufficiently large  $M$ . Choose one such  $M$ , and denote the expression (8.16) by  $C_M$ . Then relation (8.15) gives

$$\begin{aligned} e^{(-\eta + \rho)(H)} |\varphi_{\xi - i\eta}(\exp H)| &\geq C_M \prod_{\alpha \in \Sigma_0^+} (1 + |\langle \xi, \alpha_0 \rangle|)^{-\frac{m_\alpha + m_{2\alpha}}{2}} \\ &\geq C' C_M (1 + \|\xi\|)^{-\frac{\dim N}{2}} \end{aligned}$$

for all  $H \in \mathfrak{a}_M$  and all  $\xi \in \mathfrak{a}^*$ , where  $C'$  is a constant that depends only on  $\Sigma$ . Since  $\|\eta\|$  is fixed, this clearly implies the slow decrease condition (8.5) for each  $H \in \mathfrak{a}_M$ , and this in turn proves Theorem 8.1.

## 9. SURJECTIVITY IN THE RANK ONE CASE

In this section we assume that  $X = G/K$  is of rank one. The purpose of this section is to show that  $\lambda \mapsto \varphi_\lambda(h)$  is slowly decreasing, thereby proving the surjectivity of mean value operators according to Proposition 6.1.

Let  $\alpha$  and  $2\alpha$  be the positive roots and let  $p = m_\alpha$ ,  $q = m_{2\alpha}$ , respectively. In addition let  $n = \dim G/K$ . Here we note that in this case  $n = p + q + 1$ . We define a norm  $\|\cdot\|$  on  $\mathfrak{a}$  by

$$\|X\| := \left( -\frac{1}{2(p+4q)} B(X, \theta X) \right)^{\frac{1}{2}}.$$

Here  $B(\cdot, \cdot)$  and  $\theta$  denote the Killing form and the Cartan involution, respectively. We take  $H \in \mathfrak{a}$  such that  $\alpha(H) = 1$ .

Next, we identify  $\mathfrak{a}^*$  with  $\mathbb{R}$  and denote by  $\varphi_\lambda$  the zonal spherical function on  $G/K$  corresponding to  $\lambda \in \mathfrak{a}^* \cong \mathbb{R}$ . Then we have the following.

**Theorem 9.1** (Koornwinder [Koo75]). *Fix  $t > 0$  and let  $h = \exp(tH)$ . Then*

$$\begin{aligned} (9.1) \quad & \frac{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})}{2^{\frac{n-1}{2}}\Gamma(\frac{n}{2})} (\sinh t)^{n-2} (\cosh t)^{\frac{q}{2}} \varphi_\lambda(h) \\ &= \int_0^t \cos(\lambda s) (\cosh t - \cosh s)^{\frac{n-3}{2}} {}_2F_1 \left( 1 - \frac{q}{2}, \frac{q}{2}; \frac{n-1}{2}; \frac{\cosh t - \cosh s}{2 \cosh t} \right) ds. \end{aligned}$$

*Remark 9.2.* For the details of the above theorem, see also Rouvière [Rou14] p.113.

We fix  $t > 0$  and put

$$(9.2) \quad I(\lambda) := \int_0^t \cos(\lambda s) (\cosh t - \cosh s)^{\frac{n-3}{2}} {}_2F_1 \left( 1 - \frac{q}{2}, \frac{q}{2}; \frac{n-1}{2}; \frac{\cosh t - \cosh s}{2 \cosh t} \right) ds.$$

We write the holomorphic extension of  $I(\lambda)$  as  $I(\zeta)$ , ( $\zeta \in \mathbb{C}$ ). As a direct consequence of Theorem 9.1, it suffices to show that the entire function  $\mathbb{C} \ni \zeta \mapsto I(\zeta)$  is slowly decreasing. We will need to split up in two cases, namely when  $n$  is either odd or even.

**9.1. The case when  $n$  is odd.** First, we consider the case when  $n$  is an odd number. We put  $n - 3 = 2\ell$  ( $\ell \in \mathbb{Z}$ ,  $\ell \geq 0$ ).

In this case, we note that  $p = 2\ell + 2$ ,  $q = 0$  and that the hypergeometric function appearing in the integrand of  $I(\lambda)$  is just a constant function. More precisely, we have

$${}_2F_1\left(1 - \frac{q}{2}, \frac{q}{2}; \frac{n-1}{2}; \frac{\cosh t - \cosh s}{2 \cosh t}\right) = 1.$$

For a nonnegative integer  $m$ , let

$$(9.3) \quad \mathcal{I}_m(\lambda) := \int_{-t}^t \cos(\lambda s) (\cosh t - \cosh s)^m ds.$$

By the above, obviously  $I(\lambda) = \frac{1}{2}\mathcal{I}_\ell(\lambda)$ . So our objective in this subsection is to compute  $\mathcal{I}_m(\lambda)$ .

Let us put

$$f_m(s) = (\cosh t - \cosh s)^m.$$

Then we see easily that  $f_m$  satisfies

$$(9.4) \quad f_m''(s) = m^2 f_m(s) - m(2m-1) \cosh t f_{m-1}(s) + m(m-1) \sinh^2 t f_{m-2}(s), \quad (m \geq 2).$$

By integration by parts and applying (9.4), we get

$$\begin{aligned} \mathcal{I}_m(\lambda) &\equiv \int_{-t}^t \cos(\lambda s) f_m(s) ds \\ &= -\frac{1}{\lambda^2} \int_{-t}^t \cos(\lambda s) f_m''(s) ds \\ &= -\frac{1}{\lambda^2} \int_{-t}^t \cos(\lambda s) \times \\ &\quad \{m^2 f_m(s) - m(2m-1) \cosh t f_{m-1}(s) + m(m-1) \sinh^2 t f_{m-2}(s)\} ds \\ &= -\frac{m^2}{\lambda^2} \mathcal{I}_m(\lambda) - \frac{m(2m-1)}{\lambda^2} \cosh t \mathcal{I}_{m-1}(\lambda) + \frac{m(m-1)}{\lambda^2} \sinh^2 t \mathcal{I}_{m-2}(\lambda). \end{aligned}$$

Therefore, we obtain the recurrence formula

$$(9.5) \quad \mathcal{I}_m(\lambda) = \frac{1}{\lambda^2 + m^2} \{-m(2m-1) \cosh t \mathcal{I}_{m-1}(\lambda) + m(m-1) \sinh^2 t \mathcal{I}_{m-2}(\lambda)\}.$$

On the other hand, by direct computation, we have

$$(9.6) \quad \begin{aligned} \mathcal{I}_0(\lambda) &= \frac{2 \sin(\lambda t)}{\lambda}, \\ \mathcal{I}_1(\lambda) &= -\frac{2 \sinh t}{\lambda^2 + 1} \cos(\lambda t) + \frac{2 \cosh t}{(\lambda^2 + 1)} \frac{\sin(\lambda t)}{\lambda}. \end{aligned}$$

Combining (9.5) and (9.6), we have

**Theorem 9.3.** *There exist rational functions  $P_m(\lambda)$  and  $Q_m(\lambda)$  of  $\lambda$  such that*

$$\mathcal{I}_m(\lambda) = P_m(\lambda) \frac{\sin(\lambda t)}{\lambda} + Q_m(\lambda) \cos(\lambda t).$$

*Moreover, the above  $P_m(\lambda)$  and  $Q_m(\lambda)$  are real valued and smooth on  $\mathbb{R}$ .*

We are now in a position to prove that  $\mathcal{I}_m(\lambda)$  is slowly decreasing.

By Theorem 9.3,  $S_m(\lambda) := (P_m(\lambda)/\lambda)^2 + Q_m(\lambda)^2$  is a rational function which is smooth and non-negative on  $(0, \infty)$ .  $P_m$  and  $Q_m$  may have a finite number of common zeros. So if we take sufficiently large  $\xi_0(> 0)$ , then  $S_m(\lambda) > 0$  for  $\lambda > \xi_0$ . As a result, we have

$$S_m(\lambda) \geq \frac{A}{(1 + |\lambda|)^k}, \quad \text{for } \lambda \geq \xi_0,$$

for some constants  $A > 0$  and  $k$ .

For  $\xi \in \mathbb{R}$ , we define two subsets  $U_\xi$  and  $V_\xi$  of  $\mathbb{C}$  as follows.

$$\begin{aligned} U_\xi &:= \{\zeta \in \mathbb{C}; \|\zeta - \xi\| < A \log(2 + |\xi|)\}, \\ V_\xi &:= U_\xi \cap \mathbb{R}. \end{aligned}$$

Obviously,  $U_\xi \supset V_\xi$ . If necessary, we take the above  $\xi_0$  such that

$$A \log(2 + |2\xi_0|) \geq \frac{\pi}{t}, \quad 2\xi_0 - A \log(2 + |2\xi_0|) \geq \xi_0.$$

Then

$$\begin{aligned} \sup_{\zeta \in U_\xi} |\mathcal{I}_m(\zeta)| &\geq \sup_{\lambda \in V_\xi} |\mathcal{I}_m(\lambda)| \\ &= \sup_{\lambda \in V_\xi} \left| P_m(\lambda) \frac{\sin(\lambda t)}{\lambda} + Q_m(\lambda) \cos(\lambda t) \right| \\ &= \sup_{\lambda \in V_\xi} S_m(\lambda) \\ &\geq S_m(\xi) \\ &\geq \frac{A}{(1 + |\xi|)^k}, \quad \text{for } \xi \geq 2\xi_0. \end{aligned}$$

It follows easily from the above inequalities that  $\mathcal{I}_m(\zeta)$  is slowly decreasing.

**9.2. The case when  $n$  is even.** Next, we consider the case when  $n$  is an even number. Our first objective in this subsection is to give an asymptotic expansion of  $I(\lambda)$  as  $\lambda \rightarrow +\infty$ .

We put  $n - 2 = 2\ell$  ( $\ell \in \mathbb{Z}$ ,  $\ell \geq 0$ ). In this case,  $I(\lambda)$  is written in terms of Bessel functions of the first kind. More precisely, we have the following.

**Theorem 9.4.** (i)  $I(\lambda)$  has the following Bessel function series expansion.

$$(9.7) \quad I(\lambda) = \sum_{m=0}^{N-1} d_m \left( \frac{t}{\lambda} \right)^{\ell+m} J_{\ell+m}(\lambda t) + r_N(\lambda), \quad \text{where}$$

$$(9.8) \quad d_m = \frac{\pi}{2} (2\ell + 2m - 1)!! \times \sum_{j,k \geq 0, j+k=m} a_0^{\ell+k-\frac{1}{2}} b_j^{(\ell+k)} c_k.$$

The constants  $a_0, b_j^{(\ell+k)}$ , and  $c_k$  are given respectively by the expressions (10.7), (10.5), and (10.1) in Appendix A. In addition,  $L!!$  is defined as follows:  $(-1)!! = 1$ ,  $0!! = 1$ , and for any positive integer  $L$ ,  $L!!$  is the product of all the integers from 1 up to  $L$  that have the same parity as  $L$ .

(ii) The  $N$ -th remainder term  $r_N(\lambda)$  satisfies

$$|r_N(\lambda)| \leq \frac{C_N}{(|\lambda| + 1)^N}, \quad \text{for } \lambda \in \mathbb{R}.$$

We will prove the above theorem in Appendix A.

Formula (9.7) gives an explicit asymptotic expansion of  $I(\lambda)$  as  $\lambda \rightarrow \infty$ , because the asymptotic expansion of each Bessel function is well known. In fact, we have

**Theorem 9.5.** (See, for example, [GR15], §8.451, Formula 1.) If  $z \rightarrow \infty$  under the condition that  $|\arg z| < \pi$ , we have

$$\begin{aligned} J_m(z) &= \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{2}m - \frac{\pi}{4}\right) \\ &\quad \times \left\{ \sum_{k=0}^{N-1} \frac{(-1)^k \Gamma(m+2k+\frac{1}{2})}{(2k)! \Gamma(m-2k+\frac{1}{2})} \frac{1}{(2z)^{2k}} + R_N^{(1)} \right\} \\ &\quad - \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi}{2}m - \frac{\pi}{4}\right) \\ &\quad \times \left\{ \sum_{k=0}^{N-1} \frac{(-1)^k \Gamma(m+2k+\frac{3}{2})}{(2k+1)! \Gamma(m-2k-\frac{1}{2})} \frac{1}{(2z)^{2k+1}} + R_N^{(2)} \right\}, \end{aligned}$$

where the  $N$ -th remainder terms  $R_N^{(1)}$  and  $R_N^{(2)}$  satisfy

$$\begin{aligned} |R_N^{(1)}| &< \left| \frac{(-1)^N \Gamma(m+2N+\frac{1}{2})}{(2N)! \Gamma(m-2N+\frac{1}{2}) (2z)^{2N}} \right|, \quad (N > \frac{m}{2} - \frac{1}{4}) \\ |R_N^{(2)}| &< \left| \frac{(-1)^N \Gamma(m+2N+\frac{3}{2})}{(2N+1)! \Gamma(m-2N-\frac{1}{2}) (2z)^{2N+1}} \right|, \quad (N > \frac{m}{2} - \frac{3}{4}). \end{aligned}$$

Theorem 9.4 and Theorem 9.5 yield the following.

**Corollary 9.6.**  $I(\lambda)$  is written as

$$I(\lambda) = d_0 \left( \frac{t}{\lambda} \right)^\ell \times \sqrt{\frac{2}{\pi \lambda t}} \cos\left(\lambda t - \frac{\pi}{2}\ell - \frac{\pi}{4}\right) + \tilde{r}_1(\lambda),$$

where  $\tilde{r}_1(\lambda)$  satisfies

$$|\tilde{r}_1(\lambda)| \leq \frac{C}{(|\lambda| + 1)^{\ell + \frac{3}{2}}}, \quad \lambda \in \mathbb{R}.$$

Now we will prove that  $I(\lambda)$  is slowly decreasing.

For  $\xi \in \mathbb{R}$ , we define three subsets  $U_\xi$ ,  $V_\xi$ , and  $W_\xi$  of  $\mathbb{C}$  as follows.

$$U_\xi := \{\zeta \in \mathbb{C}; ||\zeta - \xi|| < A \log(2 + |\xi|)\},$$

$$V_\xi := U_\xi \cap \mathbb{R},$$

$$W_\xi := \{\lambda \in \mathbb{R}; \xi - 2\pi < \lambda t < \xi\},$$

Obviously,  $U_\xi \supset V_\xi$ . We take

$$A = \sqrt{\frac{1}{2\pi t}} d_0 t^\ell.$$

Then we see easily that

$$\sup_{\lambda \in W_\xi} \left| d_0 \left( \frac{t}{\lambda} \right)^\ell \times \sqrt{\frac{2}{\pi \lambda t}} \cos\left(\lambda t - \frac{\pi}{2}\ell - \frac{\pi}{4}\right) \right| \geq 2A|\xi|^{-\ell - \frac{1}{2}} \quad \text{for } \xi > 2\pi.$$

Next, we take sufficiently large positive constant  $\delta_0 (> 2\pi)$ . Then for  $\xi > \delta_0$ , we have  $V_\xi \supset W_\xi$ . (Take  $\delta_0$  such that  $A \log(2 + |\delta_0|) \geq 2\pi t$ .) As a result, we have

$$U_\xi \supset V_\xi \supset W_\xi, \quad \text{for } \xi > \delta_0.$$

Therefore, we have

$$\begin{aligned}
 \sup_{\zeta \in U_\xi} |I(\zeta)| &\geq \sup_{\lambda \in V_\xi} |I(\lambda)| \\
 &\geq \sup_{\lambda \in W_\xi} |I(\lambda)| \\
 &\geq \sup_{\lambda \in W_\xi} \left| d_0 \left( \frac{t}{\lambda} \right)^\ell \times \sqrt{\frac{2}{\pi \lambda t}} \cos \left( \lambda t - \frac{\pi}{2} \ell - \frac{\pi}{4} \right) \right| - \sup_{\lambda \in W_\xi} |\tilde{r}_1(\lambda)| \\
 &\geq 2A|\xi|^{-\ell-\frac{1}{2}} - \frac{C}{(|\xi|+1)^{\ell+\frac{3}{2}}} \\
 &\geq A(|\xi|+1)^{-\ell-\frac{1}{2}} + (|\xi|+1)^{-\ell-\frac{1}{2}} (A - C(|\xi|+1)^{-1}), \quad \text{for } \xi > \delta_0.
 \end{aligned}$$

Again, we take another positive constant  $\delta_1 (> \delta_0)$  such that

$$A - C(|\xi|+1)^{-1} > 0, \quad \text{for } \xi > \delta_1.$$

Then we have

$$\sup_{\zeta \in U_\xi} |I(\zeta)| \geq A(|\xi|+1)^{-\ell-\frac{1}{2}} \quad \text{for } \xi > \delta_1.$$

A similar argument holds for  $\xi < -\delta_1$ . Namely, we have

$$(9.9) \quad \sup_{\zeta \in U_\xi} |I(\zeta)| \geq A(|\xi|+1)^{-\ell-\frac{1}{2}} \quad \text{for } |\xi| > \delta_1.$$

It follows from (9.9) that  $I(\lambda)$  is slowly decreasing.

## 10. APPENDIX A: BESSEL SERIES EXPANSION

In this appendix, we will prove Theorem 9.4.

We put

$$\begin{aligned}
 F(z) &:= {}_2F_1 \left( 1 - \frac{q}{2}, \frac{q}{2}; \frac{n-1}{2}; \frac{z}{2 \cosh t} \right), \\
 Z(s) &:= \cosh t - \cosh s.
 \end{aligned}$$

Let us write the power series expansion of  $F(z)$  as

$$F(z) = \sum_{k=0}^{\infty} c_k z^k = \sum_{k=0}^{N-1} c_k z^k + F_N(z),$$

where  $F_N$  is the  $N$ -th remainder term and the  $k$ -th coefficient  $c_k$  is given by

$$(10.1) \quad c_k = \frac{(1 - \frac{q}{2})_k (\frac{q}{2})_k}{(\frac{n-1}{2})_k k! (2 \cosh t)^k}.$$

Then  $I(\lambda)$  is written as

$$\begin{aligned}
 (10.2) \quad I(\lambda) &= \sum_{k=0}^{N-1} c_k \int_0^t \cos(\lambda s) \{Z(s)\}^{\ell+k-\frac{1}{2}} ds \\
 &\quad + \int_0^t \cos(\lambda s) \{Z(s)\}^{\ell-\frac{1}{2}} F_N(Z(s)) ds.
 \end{aligned}$$

Next, we put

$$(10.3) \quad I_k(\lambda) := \int_0^t \cos(\lambda s) \{Z(s)\}^{\ell+k-\frac{1}{2}} ds$$

From now on, we will expand the right hand side of (10.3) as a series of Bessel functions. Let us define a function  $f$  by

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} z^k$$

Then we see easily that  $\cosh z = f(z^2)$ . Moreover, we have

$$\frac{f(t^2) - f(t^2 - z)}{z} = \sum_{k=0}^{\infty} a_k z^k, \quad \text{where}$$

$$a_k = \frac{(-1)^k f^{(k+1)}(t^2)}{(k+1)!}, \quad (k = 0, 1, 2, \dots).$$

Therefore, we can put

$$(10.4) \quad \left\{ \frac{f(t^2) - f(t^2 - z)}{z} \right\}^{m-\frac{1}{2}} = a_0^{m-\frac{1}{2}} \left( 1 + \frac{a_1}{a_0} z + \frac{a_2}{a_0} z^2 + \dots \right)^{m-\frac{1}{2}}$$

$$(10.5) \quad = a_0^{m-\frac{1}{2}} (1 + b_1^{(m)} z + b_2^{(m)} z^2 + \dots)$$

$$(10.6) \quad = a_0^{m-\frac{1}{2}} (1 + b_1^{(m)} z + b_2^{(m)} z^2 + \dots + b_{N-1}^{(m)} z^{N-1}) + g_{(m,N)}(z) z^N,$$

where  $g_{(m,N)}(z)$  is holomorphic near the line segment  $[0, t^2] \subset \mathbb{C}$  and where the coefficients  $b_j^{(m)}$  ( $j = 1, 2, 3, \dots$ ) are written as a polynomial of  $\frac{a_1}{a_0}, \frac{a_2}{a_0}, \dots, \frac{a_j}{a_0}$ . For example, the first three coefficients  $b_1^{(m)}, b_2^{(m)}, b_3^{(m)}$  in the above expansion are given by

$$b_1^{(m)} = (m - \frac{1}{2}) \frac{a_1}{a_0}, \quad b_2^{(m)} = (m - \frac{1}{2}) \frac{a_2}{a_0} + \binom{m - \frac{1}{2}}{2} \left( \frac{a_1}{a_0} \right)^2,$$

$$b_3^{(m)} = (m - \frac{1}{2}) \frac{a_3}{a_0} + 2 \binom{m - \frac{1}{2}}{2} \frac{a_1 a_2}{a_0^2} + \binom{m - \frac{1}{2}}{3} \left( \frac{a_1}{a_0} \right)^3.$$

Here we define  $b_0^{(m)} := 1$ . We also note that for each fixed  $t > 0$

$$(10.7) \quad a_0 = \frac{\sinh \sqrt{t}}{2\sqrt{t}} > 0.$$

By substituting  $z = t^2 - s^2$  in (10.6), we have

$$\left( \frac{\cosh t - \cosh s}{t^2 - s^2} \right)^{m-\frac{1}{2}} = a_0^{m-\frac{1}{2}} \sum_{j=0}^{N-1} b_j^{(m)} (t^2 - s^2)^j$$

$$+ g_{(m,N)}(t^2 - s^2) (t^2 - s^2)^N.$$

Thus we have

$$\{Z(s)\}^{\frac{n-3}{2}+k} \equiv \{Z(s)\}^{\ell+k-\frac{1}{2}}$$

$$\equiv (\cosh t - \cosh s)^{\ell+k-\frac{1}{2}}$$

$$= (t^2 - s^2)^{\ell+k-\frac{1}{2}} \left( \frac{\cosh t - \cosh s}{t^2 - s^2} \right)^{\ell+k-\frac{1}{2}}$$

$$= a_0^{\ell+k-\frac{1}{2}} \sum_{j=0}^{N-1} b_j^{(\ell+k)} (t^2 - s^2)^{\ell+k+j-\frac{1}{2}}$$

$$+ (t^2 - s^2)^{\ell+k+N-\frac{1}{2}} g_{(\ell+k,N)}(t^2 - s^2)$$



Therefore,  $I_k(\lambda)$  is rewritten as

$$(10.8) \quad \begin{aligned} I_k(\lambda) &= a_0^{\ell+k-\frac{1}{2}} \sum_{j=0}^{N-1} b_j^{(\ell+k)} \int_0^t \cos(\lambda s) (t^2 - s^2)^{\ell+k+j-\frac{1}{2}} ds \\ &\quad + \int_0^t \cos(\lambda s) (t^2 - s^2)^{\ell+k+N-\frac{1}{2}} g_{(\ell+k,N)}(t^2 - s^2) ds. \end{aligned}$$

Here in the R. H. S. of (10.8), we have

$$(10.9) \quad \begin{aligned} &\int_0^t \cos(\lambda s) (t^2 - s^2)^{\ell+k+j-\frac{1}{2}} ds \\ &= t^{2(\ell+k+j)} \int_0^{\frac{\pi}{2}} \cos(\lambda t \sin \theta) \cos^{2(\ell+k+j)} \theta d\theta \\ &= \frac{\pi}{2} \times (2\ell + 2k + 2j - 1)!! \times \left(\frac{t}{\lambda}\right)^{\ell+k+j} \times J_{\ell+k+j}(\lambda t). \end{aligned}$$

In the above,  $J_m(z)$  denotes the Bessel function of the first kind. We also note that in the computation of (10.9), we used the following formula for Bessel functions.

$$\int_0^{\frac{\pi}{2}} \cos(z \sin \theta) \cos^{2m} \theta d\theta = \frac{\pi}{2} \times \frac{(2m-1)!!}{z^m} \times J_m(z).$$

For details, see, for example, [GR15], §3.715, Formula 10.

Combining (10.2), (10.3), (10.8), and (10.9), we have

$$(10.10) \quad \begin{aligned} &I(\lambda) \\ &= \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} a_0^{\ell+k-\frac{1}{2}} b_j^{(\ell+k)} c_k \times \frac{\pi}{2} \times (2\ell + 2k + 2j - 1)!! \times \left(\frac{t}{\lambda}\right)^{\ell+k+j} \times J_{\ell+k+j}(\lambda t) \\ &\quad + \sum_{k=0}^{N-1} I_k^N(\lambda) + R_N(\lambda), \end{aligned}$$

where

$$(10.11) \quad I_k^N(\lambda) = \int_0^t \cos(\lambda s) (t^2 - s^2)^{\ell+k+N-\frac{1}{2}} g_{(\ell+k,N)}(t^2 - s^2) ds,$$

$$(10.12) \quad R_N(\lambda) = \int_0^t \cos(\lambda s) \{Z(s)\}^{\ell-\frac{1}{2}} F_N(Z(s)) ds.$$

Let us rewrite (10.10) as follows.

$$\begin{aligned}
 I(\lambda) &= \sum_{m=0}^{N-1} d_m \left( \frac{t}{\lambda} \right)^{\ell+m} J_{\ell+m}(\lambda t) + \widetilde{R}_N(\lambda), \\
 \text{where} \\
 d_m &= \frac{\pi}{2} (2\ell + 2m - 1)!! \times \sum_{0 \leq j, k, j+k=m} a_0^{\ell+k-\frac{1}{2}} b_j^{(\ell+k)} c_k, \\
 (10.13) \quad \widetilde{R}_N(\lambda) &= \frac{\pi}{2} \sum_{0 \leq j, k \leq N-1, N \leq j+k} a_0^{\ell+k-\frac{1}{2}} b_j^{(\ell+k)} c_k (2\ell + 2k + 2j - 1)!! \\
 &\quad \times \left( \frac{t}{\lambda} \right)^{\ell+k+j} J_{\ell+k+j}(\lambda t) \\
 &\quad + \sum_{k=0}^{N-1} I_k^N(\lambda) + R_N(\lambda)
 \end{aligned}$$

Our next objective is to estimate the remainder term  $\widetilde{R}_N(\lambda)$ . We need the following lemma.

**Lemma 10.1.** (i) *Let  $\varphi(s)$  be an even function of class  $C^{2m}$  defined on  $[-t, t]$ . We assume that*

$$(10.14) \quad \varphi(\pm t) = \varphi'(\pm t) = \varphi''(\pm t) = \dots = \varphi^{(2m-1)}(\pm t) = 0.$$

*Then we have*

$$\int_0^t \cos(\lambda s) \varphi(s) ds = \frac{(-1)^m}{\lambda^{2m}} \int_0^t \cos(\lambda s) \varphi^{(2m)}(s) ds.$$

(ii) *Moreover, we have*

$$\left| \int_0^t \cos(\lambda s) \varphi(s) ds \right| \leq \frac{C}{|\lambda|^{2m}}, \quad \text{for } \lambda \in \mathbb{R} \setminus \{0\},$$

*where the above constant  $C$  does not depend on  $\lambda$ .*

*Proof.* By repeating integral by parts  $2m$ -times, we get (i). By taking  $C = \int_0^t |\varphi^{(2m)}(s)| ds$ , we get the estimate (ii).  $\square$

We apply Lemma 10.1 to the remainder terms  $I_k^N(\lambda)$  and  $R_N(\lambda)$  by taking

$$\begin{aligned}
 \varphi(s) &= (t^2 - s^2)^{\ell+k+N-\frac{1}{2}} g_{(\ell+k, N)}(t^2 - s^2), \\
 \text{and } \varphi(s) &= \{Z(s)\}^{\ell-\frac{1}{2}} F_N(Z(s)),
 \end{aligned}$$

respectively. As a result, we have

$$(10.15) \quad |I_k^N(\lambda)| \leq \frac{C_{(k, N)}}{|\lambda|^{2\lceil \frac{\ell+k+N-1}{2} \rceil}}, \quad \text{for } \lambda \in \mathbb{R} \setminus \{0\},$$

$$(10.16) \quad \text{and } |R_N(\lambda)| \leq \frac{C_{(N)}}{|\lambda|^{2\lceil \frac{\ell+N-1}{2} \rceil}}, \quad \text{for } \lambda \in \mathbb{R} \setminus \{0\},$$

where  $[x]$  denotes the largest integer less than or equal to  $x$ . In the above estimates, the constants  $C_{(k, N)}$  and  $C_{(N)}$  do not depend on  $\lambda$ . Next, we go into the estimate of Bessel functions. By the definition of the Bessel function

$$J_m(z) = \frac{1}{\pi} \int_0^\pi \cos(m\theta - z \sin \theta) d\theta,$$

we have

$$(10.17) \quad J_m(\lambda t) \leq 1, \quad \text{for } \lambda \in \mathbb{R}.$$

By (10.15), (10.16), and (10.17), we have

$$(10.18) \quad |\widetilde{R_N}(\lambda)| \leq \frac{\widetilde{C_N}}{(|\lambda| + 1)^{2[\frac{N-1}{2}]}}, \quad \text{for } \lambda \in \mathbb{R},$$

if we take some suitable constant  $\widetilde{C_N}$ .

Here we replace  $N$  by  $2N + 1$  in (10.13). Then we have

$$(10.19) \quad I(\lambda) = \sum_{m=0}^{N-1} d_m \left( \frac{t}{\lambda} \right)^{\ell+m} J_{\ell+m}(\lambda t) + r_N(\lambda), \quad \text{where}$$

$$(10.20) \quad r_N(\lambda) = \sum_{m=N}^{2N} d_m \left( \frac{t}{\lambda} \right)^{\ell+m} J_{\ell+m}(\lambda t) + \widetilde{R_{2N+1}}(\lambda).$$

We apply (10.17) and (10.18) to R.H.S of (10.20). Then we see easily that there exists a positive constant  $C_N$  independent of  $\lambda$  such that

$$|r_N(\lambda)| \leq \frac{C_N}{(|\lambda| + 1)^N}, \quad \text{for } \lambda \in \mathbb{R}.$$

This finishes the proof of Theorem 9.4.

## 11. APPENDIX B: A SMOOTHNESS RESULT

In this section we will prove a smoothness result for quotients that is needed to complete the proof of Theorem 5.1.

We first introduce some notation which we will use below. For any  $z \in \mathbb{C}$  and  $r > 0$ , let  $D_r(z) = \{\zeta \in \mathbb{C} : |\zeta - z| \leq r\}$ , let  $\overline{D_r}(z)$  be its closure, and let  $C_r(z)$  be the circle  $\{\zeta \in \mathbb{C} : |\zeta - z| = r\}$ . If  $z = 0$ , we will just use the notation  $D_r$  and  $C_r$  in place of  $D_r(0)$  and  $C_r(0)$ .

**Proposition 11.1.** *Let  $U \subset \mathbb{R}^m$  and  $V \subset \mathbb{C}^n$  be open sets, and let  $F(x, z)$  be  $C^\infty$  on  $U \times V$  and holomorphic in  $z$  for each fixed  $x$ . Suppose that  $g(z)$  is a nonzero holomorphic function on  $V$  such that  $F(x, z)/g(z)$  is holomorphic for each  $x$ . Then  $F(x, z)/g(z)$  is  $C^\infty$  on  $U \times V$ .*

*Proof.* It suffices to prove that  $F/g$  is smooth on a neighborhood of each point  $(x_0, z_0) \in U \times V$  for which  $g(z_0) = 0$ . For simplicity, we can assume that  $x_0 = 0$  and  $z_0 = 0$  and then by shrinking  $U$  and  $V$ , we can assume that  $U$  and  $V$  are balls centered at the origins in  $\mathbb{R}^m$  and  $\mathbb{C}^n$ , respectively. We can also assume that  $F$  is not identically 0.

Let  $\mathcal{V} \subset V$  be the zero locus  $\mathcal{V} = \{z \in V : g(z) = 0\}$ . Since  $g$  is not identically 0, it is not identically 0 on some complex line  $\ell$  through the origin in  $V$ ; applying a linear operator on  $\mathbb{C}^n$ , we can assume that  $\ell = \{z \in V : z_2 = \dots = z_n = 0\}$ . (If  $n = 1$ , we have  $\ell = V$ .)

Since we're trying to prove the smoothness of  $F/g$  on a neighborhood of  $(0, 0)$ , we can conveniently shrink  $V$  even further and assume that  $V$  is an open polydisk  $D_{r_1} \times \dots \times D_{r_n}$  in  $\mathbb{C}^n$ , so  $\ell = D_{r_1} \times \{0\} \times \dots \times \{0\}$ .

Now  $\ell \cap \mathcal{V}$  consists of isolated points, so in particular there is closed disk  $\overline{D_{s_1}}$  (with positive radius  $s_1$ ) inside  $D_{r_1}$  such that  $\mathcal{V} \cap (D_{s_1} \times \{0\} \times \dots \times \{0\}) = (0, \dots, 0)$ .

Since  $C_{s_1} \times \{0\} \times \dots \times \{0\}$  is a compact set disjoint from the (relatively) closed set  $\mathcal{V}$ , there is a tube  $C_{s_1} \times \overline{D_{s_2}} \times \dots \times \overline{D_{s_n}}$  in  $V$  disjoint from  $\mathcal{V}$ . Then in particular,  $C_{s_1} \times C_{s_2} \times \dots \times C_{s_n}$  is disjoint from  $\mathcal{V}$ . Let  $V' = D_{s_1} \times \dots \times D_{s_n}$ .

Now for any  $(x, z) \in U \times V'$ , we have

$$\begin{aligned} \frac{F(x, z)}{g(z)} &= \frac{1}{(2\pi i)^n} \int_{C_{s_1}} \cdots \int_{C_{s_n}} \frac{F(x, \zeta_1, \dots, \zeta_n)/g(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} d\zeta_n \cdots d\zeta_1. \end{aligned}$$

Since  $|g(\zeta_1, \dots, \zeta_n)|$  is bounded below by a fixed positive constant on the compact set  $C_{s_1} \times \cdots \times C_{s_n}$ , we can apply the usual arguments for differentiating inside the integral to conclude that if  $\alpha \in (\mathbb{Z}^+)^m$  and  $\beta = (\beta_1, \dots, \beta_n) \in (\mathbb{Z}^+)^n$ , then

$$\begin{aligned} D_x^\alpha (\partial_{z_1}^{\beta_1}) \cdots (\partial_{z_n}^{\beta_n}) \left( \frac{F(x, z)}{g(z)} \right) &= \frac{\beta_1! \cdots \beta_n!}{(2\pi i)^n} \int_{C_{s_1}} \cdots \int_{C_{s_n}} \frac{D_x^\alpha F(x, \zeta_1, \dots, \zeta_n)/g(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1)^{\beta_1+1} \cdots (\zeta_n - z_n)^{\beta_n+1}} d\zeta_n \cdots d\zeta_1 \end{aligned}$$

for all  $(x, z) \in U \times V'$ . This of course proves the lemma.  $\square$

The following example shows that analyticity is needed in the second argument. Let  $m = n = 1$  and define the function  $g(y)$  on  $\mathbb{R}$  by

$$g(y) = \begin{cases} e^{-1/y^2} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0. \end{cases}$$

Then the function on  $\mathbb{R}^2$  given by

$$F(x, y) = \begin{cases} xg(y)/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

is smooth, but  $F(x, y)/g(y)$  does not extend to a continuous function on  $\mathbb{R}^2$ .

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